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U N I V E R Z I T E T U N O V O M S A D U

PRIRODNO-MATEMATIČKI FAKULTET

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PRIMENA NOVOG BOZONSKOG FORMALIZMA U NEKIM PROBLEMIMA KVANTNE TEORIJE ČVRSTOG STANJA

- doktorska disertacija -

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Prijatna mi je dužnost da se zahvalim kolegama iz Laboratorije za teorijsku fiziku Instituta za nuklearne nauke "Boris Kidrič" u Vinči i kolegama iz Zavoda za fiziku i matematiku i Prirodno matematičkog fakulteta u Novom Sadu,za prijatnu i stimulativnu atmosferu u toku rada na tezi.

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J. Vujaklija

S A D R Ž A J

U V O D

U ovoj disertaciji biće ispitan i primenjen novi bozonski formalizam za analizu nelinearnih efekata u kristalima, poznat pod imenom egzaktna bozonska reprezentacija spinskih operatora. Glavni cilj teze je da se ova reprezentacija testira na problemima kvantne teorije magnetizma. U tom cilju izabrana su neka pitanja kao reprezenti tri grupe problema. Prva uključuje probleme koji su ranije uspešno razmatrani pomoću drugih formalizama / takav je problem niskotemperaturske analize Heisenberg -ovog feromagneta sa spinom 1/2 /,ili probleme koji su analizirani nedovoljno korektno / kao npr. problem vezanih stanja u Heisenberg-ovom feromagnetu/. Pošto je ove probleme uglavnom moguće egzaktno rešiti, želim da ispitam da li se pomoću novog formalizma dobijaju poznati rezultati, i ujedno utvrdim, poređenjem sa ranijim prilazima,njegove metodološke prednosti i nedostatke.

Druga grupa obuhvata probleme na kojima se radi u svetu i koji su zato aktuelni /kao što je problem Bose kondenzacije u Heisenberg-ovom feromagnetu sa spinom 1/2 /. Konačnu reč o rezultatima i vrednosti novog metoda daće u ovom slučaju,kao i u nekim drugim problemima koji su razmatrani u grupi dr. B.Tošića u kojoj radim,sam eksperiment.

Treća grupa obuhvata matematičke probleme u vezi sa egzaktnom bozonskom reprezentacijom spinskih operatora.U disertaciji će biti razmotren problem hermiticiteta Pauli-Hamiltonijana u egzaktnoj bozonskoj reprezentaciji.

Teza je podeljena na tri glave.

U prvoj glavi razmatraju se dinamičke karakteristike Heisenberg-ovog feromagneta sa spinom $S= 1/2$. na niskim temperaturama. Ovo uključuje izračunavanje spektra elementarnih eksitacija kao i teoriju Bose kondenzacije u feromagnetu.

U drugoj glavi tretira se problem vezanih stanja u Heisenberg-ovom feromagnetu, a u trećoj se egzaktno dokazuje da su Pauli Hamiltonijani u egzaktnoj bozonskoj slici hermitski.

Na kraju teze dat je zaključak koji sumira glavne rezultate i procenjuje vrednost razvijenih metoda.

Glava prva

Dinamičke karakteristike Heisenberg-ovog feromagneta sa spinom $S=\frac{1}{2}$ na niskim temperaturama.

A. Razlozi, ciljevi i metodi

Teorijski opis feromagnetnih osobina materije je jedan od onih problema koji već skoro dva veka kontinuirano zaokupljaju pažnju fizičara. Ni kvantna fizika zato nije mogla da mimoiđe ovaj problem. Njen najprostiji model za opisivanje feromagnetizma jeste Heisenberg-ov izotropni model u kome se polazi od ideje izmene kao fundamentalne osobine onih interakcija u kristalu koje definišu njegova energijska stanja.

Hamiltonijan ovog modela sadrži spinske operatore atoma koji, međutim, nemaju ni bozonske, ni fermionske komutacione relacije, pa se kao prvi problem pri izračunavanju termodinamičkih veličina pojavljuje problem izbora statistike. Pored toga u kvantno - mehaničkim izračunavanjima veoma često je potrebno preći iz prostora direktne u prostor recipročne rešetke, a komutacione relacije za spinske operatore nisu invarijante u odnosu na odgovarajuću Fourier-ovu transformaciju, što može da prouzrokuje grešku koju je nemoguće kontrolisati. Zato je neophodno da se spinski operatori izraze preko operatora koji se transformišu kanonički i imaju određenu statistiku i u impulsnom prostoru. Kao što je poznato to su Bose i Fermi operatori.

Rešenje problema inicirano je pionirskim radom Bloch-a.[1] On je pokazao da elementarne ekscitacije u odnosu na osnovno stanje Heisenberg-ovog feromagneta /u kojem su spinovi svih atoma "gore"/

jesu stanja sa po jednim spinom suprotne orijentacije i da su ovakva stanja funkcije talasnog vektora $\vec{k}$. Predpostavljajući da spinski talasi kako je nazvao ove elementarne ekscitacije, formiraju gas neinteragujućih Bose čestica energije srazmerne sa k^2 za malo $k=|\vec{k}|$, Bloch je zaključio da magnetizacija opada sa temperaturom po "$T^{3/2}$" zakonu. Međutim sa porastom temperature povećava se broj ekscitiranih spinskih talasa, a time i verovatnoća za interakciju spinskih talasa. Matematički, Bloch je spinske operatore predstavio preko bozonskih pomoću linearne, harmoniske aproksimacije, te se zato efekti interakcije elementarnih ekscitacija označavaju kao nelinearni ili anharmonijski.

Prvi pokušaj analize nelinearnih efekata učinjen je od strane Holstein-a u Primakoff-a[2] /H-P/ koji su spinske operatore predstavili pomoću bozonskih i analizirali ekvivalentni bozonski sistem. Ako se u spinskom Hamiltonijanu samo članovi drugog reda po operatorima kreacije S^+ odnosno anihilacije S^- spinskog talasa izraze pomoću H-P formula, tada se pored kvadratnih članova po Bose operatorima dobijaju i članovi višeg reda. Ovi su članovi posledica razlike u komutacionim relacijama za spinske i bozonske operatore /razlike u kinematici spinskih i Bose-operatora/ te se odgovarajući deo Hamiltonijana naziva kinematičkim. Slično, bozonske članove istog reda /višeg od drugog/ kao spinski od kojih nastaju u Bloch-ovoj oproklimaciji, nazivamo dinamičkim, a dodatne bozonske članove još višeg reda dinamičko-kinematičnim.

Istraživanje nelinearnih efekata koji su posebno važni u okolini temperature prelaza, zahteva poznavanje tačnih izraza za Hamiltonijane kinematičke i dinamičko-kinematičke interakcije. H-P reprezentacija, međutim, može da se koristi jedino na dovoljno niskim temperaturama, tj. za niske koncentracije elementarnih ekscitacija kada su nelinearni efekti zanemarljivi. Ovo je posledica činjenice da su spinske komutacione relacije u ovoj reprezentaciji zadovoljene samo u onom delu Hilbert-ovog prostora koji se sastoji od bozonskih vektora stanja čiji /bozonski/ populacioni broj N ne prelazi 2S /S efektivni spin atoma/. Greška pri korišćenju N-P reprezentacije dolazi baš od doprinosa takozvanih "nefizičkih" stanja sa $N > 2S$ koji su tim veći što je temperatura prelaza veća.

Problem korektnog opisa anharmonijskih efekata je dalje rešavan od strane niza autora koji su na različite načine dobijali različite popravke za magnetizaciju [3] ali je fundamentalnu teoriju razvio tek Dyson [4].

Pomoću dosta glomaznog matematičkog aparata on je uspeo da pokaže da su doprinosi koji termodinamičkim veličinama dolaze od kinematičke interakcije /a odgovaraju procesima rasejanja spinskih talasa, jednog na drugom/ reda $e^{-a\frac{T_c}{T}}$ /gde je T_c - temperatura prelaza, a - konstanta reda jedinice, T - temperatura feromagneta/, dok je za popravku najnižeg reda koju daje dinamička interakcija dobio T^4. Kako je Dyson-ova procedura korektna može se reći da je njegova reprezentacija, mada nehermitska, efektivno dobra, jer bi članovi, koji bi je dopunili tako da postane herontska dali popravke koje odgovaraju procesima sudara tri elementarne ekscitacije na jednom čvoru.

Dyson-ovi članci nisu, međutim, označili i kraj reda na ovom problemu. Budući da je njegovo rešenje korektno, napori su koncentrisani na pokušaje da se Dyson-ovi rezultati dobijaju neposredno sa spinskim operatorima, što je uspelo, ali samo za spin $S > \frac{1}{2}$, Tahir Kheli-u i der-Haar-u [5].

S jedne strane, postoje radovi[6] u kojima se dobija pozitivna korekcija reda T^3 u izrazu za magnetizaciju, dok je u drugima[7] metodom spinskih Green-ovih funkcija dobijen Dyson-ov rezultat ali bez ubedljivog dokaza da je dekuplovanje sistema Green-ovih funkcija, na osnovu kojeg je rezultat dobijen, zaista korektno.

Iz ovih razloga mi smo postavili za zadatak da pokažemo da se Dyson-ovi rezultati za zakon disperzije i magnetizaciju Heisenberg-ovog feromagneta sa spisnom $S=\frac{1}{2}$ dobijaju automatski, bez dodatnih dokaza koji zahtevaju glomazni matematički aparat /kao kod Dysona/, ako se koristi egzaktna bozonska reprezentacija Agranoviča i Tošića[8]. Pored čisto metodoloških prednosti cele procedure koje se očituju u sažetosti i elegenciji kao i mogućnosti da lako kontroliše tačnost apraksimacija, novi metod ima za posledicu drugačiju fizičku sliku procesa u feromagnetu kao što će biti izloženo u odeljku B ove glave.

Drugi problem koji je u ovoj tezi razmotren jeste problem Bose kondenzacije u Heisenberg-ovom feromagnetu sa spinom $S=\frac{1}{2}$. Ovaj problem predstavlja prirodnu ekstenziju problema nelinearnih efekata u feromagnetu i to u jednom specijalnom slučaju tj. kada je feromagnet pod takvim uslovima da je Bose kondenzacija verovatna.

Cilj je da se i na ovom problemu testipa egzaktna bozonska reprezentacija Agranoviča i Tošića, tj. da se pomoću nje ispitaju termodinamičke karakteristike feromagneta u uslovima Bose kondenzije.

Zbog toga je neophodno da se prvo proanaliziraju mogući procesi na δ - potencijalu kako za feromagnet u jakom tako i za feromagnet u slabom magnetnom polju, a zatim detaljno istraži spektar elementarnih ekscitacija feromagneta i to u oba slučaja.

U oba rada priložena u ovoj glavi koriščena je tehnika Greenovih funkcija, u prvom standardna tehnika, a u drugom tehnika koju je za sisteme za kondenzatom razredio Beljajev[9].

B. Rezultati

U prvom radu[10], priloženom u ovoj glavi razvijen je jedan novi metod niskotemperaturske analize termodinamičkih karakteristika feromagneta sa spinom $S=\frac{1}{2}$ koja se sastoji u sledećem:

1. Heisenberg-ove jednačine kretanja koje definišu Green-ovu funkciju sistema $\langle\langle S^{+}_{\vec{k}}(t) | S^{-}_{\vec{k}}(t') \rangle\rangle$ postavljene su odmah u impulsnom prostoru budući da je energija elementarnih ekscitacija određena polom Green-ove funkcije $G_{\vec{k}}(E) = \langle\langle S^{+}_{\vec{k}} | S^{-}_{\vec{k}} \rangle\rangle$ koja predstavlja energijsko - vremenski Fourier-ov transform prve funkcije.

2. Pošto se spinski operatori ne transformišu kanonički pri prelazu iz prostora direktne u prostor recipročne rešetke, što implicira opasnost unošenja grešaka (koje je nemoguće kontrolisati) u sve fizičke rezultate, spinski operatori su predstavljeni pomoću egzaktne bozonske reprezentacije spinskih operatora pa je dalja analiza vršena pomoću Bose operatora. Treba još napomenuti je primenjena procedura dekuplovanja Green-ovih funkcija sistema mnogo opštija od uobičajene.

3. Dobijeni spektar elementarnih ekscitacija[10] /formula/ 2.24//:

$$E_{\vec{k}} = \Delta - \frac{1}{2}J_{\vec{k}} + \frac{1}{N}\sum_{\vec{q}}\left(J_{\vec{k}} + J_{\vec{q}} - J_{\vec{k}-\vec{q}} - J_0\right)\langle B^{+}_{\vec{q}} B_{\vec{q}}\rangle. \qquad /1.1/$$

gde je: $$\Delta = \mu\mathcal{H} + \frac{1}{2}J_0 \qquad /1.2/$$

μ - magnetni moment atoma

$\mathcal{H}$ - spoljašnje magnetno polje

$J_0 = 6I$; I - integral izmene $\qquad$ /1.3/

$$J_{\vec{k}} = I \sum_{\vec{\lambda}} \cos \vec{k}\vec{\lambda} \qquad /1.4/$$

$$\langle B^{+}_{\vec{q}} B_{\vec{q}} \rangle_{0} = \frac{1}{N} \sum_{\vec{k}} \left(e^{\frac{E^{(0)}_{\vec{k}}}{kT}} - 1 \right)^{-1} \qquad /1.5/$$

$$E^{(0)}_{\vec{k}} = \Delta - \frac{1}{2} J_{\vec{k}} \qquad /1.6/$$

$\vec{k}$ - talasni vektor

i magnetizacija[10)] Haisenberg-ovog feromagneta sa spinom $S= \frac{1}{2}$ /formula /3.5//

$$\sigma_{1/2} = 1 - 2Z_{3/2}(\alpha)\tau^{3/2} - \frac{3\pi}{2} Z_{5/2}(\alpha)\tau^{5/2} - \frac{33\pi^{2}}{16} Z_{7/2}(\alpha)\tau^{7/2} - 6\pi Z_{3/2}(\alpha)\tau^{4} + O(\tau^{9/2}) \qquad /1.7/$$

$$Z_{p}(\alpha) = \sum_{n=1} n^{-p} e^{-n\alpha}; \quad \alpha = \frac{\mu \mathcal{H}}{kT}; \quad \tau = \frac{kT}{2\pi I}$$

u potpunosti se poklapaju sa poznatim Dyson-ovim rezultatima[4)]. Na ovaj način razvijena je metoda koja, za razliku od Dyson-ove[4)], nema potrebe za "komplikovanom matematičkom mašinerijom" neophodnom da se dokaže da procesi sudara dveju ekscitacija na jednom čvoru daju eksponcijalno male doprinose svim termodinamičkim veličinama, jer nova metoda, budući da ovo inkorporira, automatski vodi korektnim rezultatima. Kako je ceo postupak, koji ne traži nikakve dodatne dokaze, mnogo "čistiji", jednostavniji i konsistentiji, činjenicu da su dobijeni Dyson-ovi rezultati smatramo kao veoma značajnu za ispravnost samog metoda.

4. Međutim, novi metod, razvijen u radu[10], implicira, kako je to utvrđeno, sasvim novi opis fizičkih procesa u feromagnetu: Elementarne ekscitacije u idealnom feromagnetu dobro su opisane pomoću bozonskih kvazičestica koje međutim, za razliku od prеđašnjih priloza, interaguju tako da je njihov populacioni broj[10] /formula /2.28 //

$$\langle B_{\vec{n}} B_{\vec{n}} \rangle = \frac{1}{N}\sum_{\vec{k}}\left[\exp\left(\frac{E_{\vec{k}}}{kT}\right)-1\right]^{-1} + \frac{2}{N}\sum_{\vec{k}}\left[\exp\left(\frac{E_{\vec{k}}}{kT}\right)-1\right]^{-1}\frac{1}{N}\sum_{\vec{q}}\left[\exp\left(\frac{E^{(0)}_{\vec{q}}}{kT}\right)-1\right]^{-1} \quad .../1.8/$$

veći nego kod sistema neinteragujućih bozona /1.5/ Izmena statistike bozonskih kvazičestica usled njihovih interakcija posledica je, u krajnjoj liniji, razvijenog metoda čija je bitna tačka korišćenje egzaktne bozonske reprezentacije. Obzirom na to da je egzaktna bozonska reprezentacija dala dobre rezultate u nelinearnoj optici [11], teoriji supraprovodljivosti[12], i omogućila generalizaciju Bogoljubovljevog metoda približne druge kvantizacije[13] kao i da daje korektne rezultate u feromagnetizmu[10], [14] i feroelektricitetu[15], naš je zaključak da moramo imati poverenja u sve što ova reprezentacija, u okvirima korektno razvijenog i primenjenog metoda, implicira, bar sve datle dok ona ili njene posledice ne budu osporene eksperimentom.

Prednost razvijenog metoda, pored matematičke jednostavnosti i fizičke konsistentnosti, predstavlja takođe mogućnost da se precizno kontroliše tačnost korišćenih aproksimacija. Svi rezultati tačni su do na članove proporcionalne kvadratu koncentracije nein-

teragujućih bozona kao što je to utvrđeno u samom članku[10].
Problemi koji su razmatrani u drugom radu[16] priloženom u ovoj glavi prirodno se nadovezuju na dosadašnja razmatranja. Rezime tih istraživanja predstavljaju sledeći zaključci:

5. Pošavši od bozonskog ekvivalenta Pauli - Hamiltonijana Heisenberg-ovog feromagneta izvršena je analiza mogućih procesa na ζ - potencijalu kako u slučaju kada se feromagnet nalazi u jakom spoljašnjem magnetnom polju tako i za slabo magnetno polje. Primenom teorije rasejanja na ζ -potencijalu[17] pokazano je da Bose kondenzacija kao statistička fluktuacija trajanja reda $t < 10^{-5}$ s /dok je vreme potrebno da se dostigne termodinamička ravnoteža, usled procesa rasejanja kvazičestica, reda $t_c \sim 10^{-8} - 10^{-10}$ s / predstavlja u slučaju jakih magnetnih polja deminantan proces. U slučaju slabih polja u feromagnetu se, prema našoj teoriji, odigravaju dva konkurentna procesa: kondenzacija i slepljivanje dva bozona u novu kvazičesticu koja ima dimenzije mnogo veće od dimenzije inicijalnih bozona i koja je zato lokalizovana. Posledica ovog je da zakon disperzije koji odgovara ovim novim kvazičesticama ne zavisi od talasnog broja $|\vec{k}|$ odnosno da je populacioni broj a s njim i doprinosi termodinamičkim veličinama srazmerni sa $\exp(-\frac{const}{kT})$.
Na ovaj način je potvrđen Dyson-ov rezultat da procesi rasejanja kvazičestica u Heisenberg-ovom feromagnetu u slabom magnetnom polju daju pri niskim temperaturama eksponencijalno male doprinose.

6. Analiza spektra elementarnih ekscitacija u feromagnetu izvršena

je pomoću tehnike Grinovih funkcija koju je, za sisteme sa kondenzatom, razvio Beljajev[9]. Cela procedura je detaljno izložena u radu[16], a rezultati su sledeći: U slučaju jakog polja ako je intenzitet talasnog vektora $|\vec{k}|$ nekondenzovanih bozona mnogo manji od kvadratnog korena iz koncentracije kondenzovanih bozona n_0, spektar linearno zavisi od $|\vec{k}|$ tj., dobijamo "akustični" zakon disperzije[16] /formula /3.15/:

$$E(\vec{k}) - \mu\mathcal{H} = \sqrt{I\beta}\,|\vec{k}|$$

$$\beta = (2\pi I + J_0)\,n_0\ ; \quad n_0 = \frac{N_0}{N} \qquad /1.9/$$

N_0 - broj bozona u kondenzatu

N - broj atoma u kristalu

Ovo se moglo i očekivati jer kada je talasni broj mali, baš kao kod zvučnih talasa, svi bozoni su praktično u kondenzatu. Ako je pak $|\vec{k}| \gg \sqrt{n_0}$ dobija se uobičajeni kvadratni zakon disperzije:

$$E(\vec{k}) \propto \frac{1}{2} I |\vec{k}|^2 \qquad /1.10/$$

Kao što smo već pomenuli, u slučaju slabog polja procesu kondenzacije konkuriše proces vezivanja dva bozona na jednom čvoru u novu kvazičesticu. Ovaj proces se odvija istom brzinom kao kondenzacija pa formiranje kondenzazata nije više tako sigurno kao u slučaju jakih magnetnih polja. Ako se, ipak, predpostavi da se kondenzat formira dobija se, uz odbacivanje svih grafova koji odgovaraju pojavi dva bozona na jednom čvoru /jer oni daju eksponencijalno male doprinose termodinamičkim veličinama/ interesantan rezultat: spektar elementarnih ekscitacija i magnetizacija feromagneta poklapaju se sa dobro poznatim Dyson-ovim formulama /1.1/ odnosno /1.7/. Prema tome, za slaba magnetna polja osnovni rezultati ne zavise od egzistencije kondenzata.

ON THE LOW-TEMPERATURE ANALYSIS OF HEISENBERG FERROMAGNET WITH SPIN $S = \frac{1}{2}$

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1. INTRODUCTION

The main problem of the theoretical description of an ideal ferromagnet is how to include correctly the effects of the interaction between the elementary excitations. As it was mentioned in [1] , the problem was considered by many authors obtaining different results in the various treatments. In the papers [1,2] the fundamental theory was set by Dyson.

The crucial point of Dyson's theory, as it turned out, is the proof that the processes in the ferromagnet arising from the collisions of two excitations at one lattice point /the scattering of two spin waves, one at the other/ give the contributions proportional to $\exp(-\frac{const}{kT})$ to all thermodynamical characteristics of the ferromagnet. That is the reason why these processes may be neglected at the low temperatures.

In this paper, however, it will be demonstrated how the Dyson's results for the energy spectrum and magnetization of an ideal ferromagnet with spin $S = \frac{1}{2}$ are obtainable in a straightforward way without the additional proofs demanding the complicated mathematical machinery if one uses the exact boson representation of the spin operators from [3]. The whole procedure is carried out in a more consistent way due to the fact that the representation [3] is the exact and the hermitic one. Besides, this is the only representation enabling us to control easily the exactness of the approximation. As it will be seen in the conclusion of this paper, if we consider only two-particle processes in a system the obtained results are exact up to $T^{9/2}$.

2. THE ENERGY SPECTRUM OF ELEMENTARY EXCITATIONS IN A FERROMAGNET WITH EFFECTIVE SPIN $S = \frac{1}{2}$

The spin operator representation of a simple cubic crystal has the following form in the nearest-neighbours approximation:

$$H = H_0 + \Delta \sum_{\vec{n}} \left(\frac{1}{2} - S^z_{\vec{n}}\right) - \frac{I}{2} \sum_{\vec{n},\vec{\lambda}} S^-_{\vec{n}} S^+_{\vec{n}+\vec{\lambda}} - \frac{I}{2} \sum_{\vec{n},\vec{\lambda}} \left(\frac{1}{2} - S^z_{\vec{n}}\right)\left(\frac{1}{2} - S^z_{\vec{n}+\vec{\lambda}}\right) \qquad (2.$$

where

$$H_0 = \frac{1}{2} N \mu \mathcal{H} - \frac{1}{8} N J_0 \,; \quad \Delta = \mu \mathcal{H} + \frac{1}{2} J_0$$

N - the number of atoms in the crystal,

μ - the magnetic moment of an atom,

$\mathcal{H}$ - the external magnetic field,

I - the exchange integral for the nearest neighbours,

$J_0 = 6I$

$\vec{\lambda}$ - the vector connecting the nearest neighbours and

$S^z_{\vec{n}}, S^{\pm}_{\vec{n}} = S^x_{\vec{n}} \pm i S^y_{\vec{n}}$ - the spin operators satisfying the commutation relations:

$$\left.\begin{array}{c}\left[S^+_{\vec{n}}, S^-_{\vec{m}}\right] = 2S^z_{\vec{n}}\delta_{\vec{n}\vec{m}} \\ (S^+_{\vec{n}})^2 = (S^-_{\vec{n}})^2 = 0 \\ \left\{S^+_{\vec{n}}, S^-_{\vec{n}}\right\} = 1\end{array}\right\} \qquad /2.2/$$

Since the relations /2.2/ do not transform cannonically under the Fourier transformation, we shall use the exact boson representation of the spin operators from [3] :

$$S^+_{\vec{n}} = \left[\sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(1+\nu)!}(B^+_{\vec{n}})^\nu B^\nu_{\vec{n}}\right]^{\frac{1}{2}} B_{\vec{n}}\ ; \quad \frac{1}{2} - S^z_{\vec{n}} = \sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(1+\nu)!}(B^+_{\vec{n}})^{\nu+1} B^{\nu+1}_{\vec{n}} \qquad (2.3)$$

In the low-temperature analysis of the ferromagnet which has to be accurate up to the order of T^4 , it is enough, as we shall see, to take into account only the two-particle processes in the equivalent boson system. Therefore, we may use the approximate formulae following from /2.3/ :

$$S^+_{\vec{n}} = B_{\vec{n}} - B^+_{\vec{n}}B_{\vec{n}}B_{\vec{n}}\ ; \quad S^-_{\vec{n}} = B^+_{\vec{n}} - B^+_{\vec{n}}B^+_{\vec{n}}B_{\vec{n}}\ ; \quad \frac{1}{2} - S^z_{\vec{n}} = B^+_{\vec{n}}B_{\vec{n}} - (B^+_{\vec{n}})^2 B^2_{\vec{n}} \qquad (2.4)$$

After the Fourier transformations of /2.4/ we get:

$$S^+_{\vec{k}} = B_{\vec{k}} - \frac{1}{N}\sum_{\vec{q}_1,\vec{q}_2} B^+_{\vec{q}_1} B_{\vec{q}_2} B_{\vec{k}+\vec{q}_1-\vec{q}_2}\ ; \quad S^-_{\vec{k}} = B^+_{\vec{k}} - \frac{1}{N}\sum_{\vec{q}_1,\vec{q}_2} B^+_{\vec{k}+\vec{q}_1-\vec{q}_2} B^+_{\vec{q}_2} B_{\vec{q}_1} \qquad (2.5)$$

If we introduce /2.5/ into /2.1/ it follows:

$$H = H_2 + H'_4 + H''_4 \qquad (2.6)$$

where:

$$H_2 = \sum_{\vec{k}_1}\left(\Delta - \frac{1}{2}J_{\vec{k}_1}\right) B^+_{\vec{k}_1} B_{\vec{k}_1} \qquad (2.7)$$

$$H'_4 = \frac{1}{N}\sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3}\left(-\Delta + \frac{1}{2}J_{\vec{k}_1} + \frac{1}{2}J_{\vec{k}_3}\right) B^+_{\vec{k}_1} B^+_{\vec{k}_2} B_{\vec{k}_3} B_{\vec{k}_1+\vec{k}_2-\vec{k}_3} \qquad (2.8)$$

$$H''_4 = \frac{1}{N}\sum_{\vec{k}_1,\vec{k}_2,\vec{k}_3} J_{\vec{k}_1-\vec{k}_3} B^+_{\vec{k}_1} B^+_{\vec{k}_2} B_{\vec{k}_3} B_{\vec{k}_1+\vec{k}_2-\vec{k}_3} \qquad (2.9)$$

$$J_{\vec{k}} = I\sum_{\vec{\lambda}} \cos \vec{k}\vec{\lambda}$$

The coefficient $E_{\vec{k}}=\Delta-\frac{1}{2}J_{\vec{k}}$ in the quadratic part /2.7/ of the Hamiltonian /2.6/ expresses Bloch's dispersion low. Due to the different commutation relations of the spin and boson operators we have the term of the kinematical interaction H_4' while H_4'' corresponds to the dynamical interaction.

In order to find the energy spectrum of elementary excitations in the system defined by /2.6/ we shall use the well-known method of time-temperature Green's functions developed in [4,5]. The energy E of the elementary excitations has to be found as the pole of the Green's function

$$G_{\vec{k}}(E)=\langle\langle S^+_{\vec{k}}|S^-_{\vec{k}}\rangle\rangle \tag{2.10}$$

representing the Fourier transform of the Green's function $\langle\langle S^+_{\vec{k}}(t)|S^-_{\vec{k}}(t')\rangle\rangle$. Deriving the function $\langle\langle A(t)|B(t')\rangle\rangle$ of arguments t and t' we get after the time-energy Fourier transformations, respectively:

$$E\langle\langle A|B\rangle\rangle=\frac{i}{2\pi}\langle[A,B]\rangle+\langle\langle[A,H]|B\rangle\rangle \tag{2.11}$$

$$E\langle\langle A|B\rangle\rangle=\frac{i}{2\pi}\langle[A,B]\rangle-\langle\langle A|[B,H]\rangle\rangle \tag{2.12}$$

where A, B are the arbitrary operators, H - the Hamiltonian of the system, while E represents its energy.

On account of the formulae /2.5/-/2.12/ we obtain the following set of equations:

$$\left(E-\Delta+\frac{1}{2}J_{\vec{k}}\right)\langle\langle S^+_{\vec{k}}|S^-_{\vec{k}}\rangle\rangle=\frac{i}{2\pi}(1-2C)+\frac{1}{2N}\sum_{\vec{k}_1,\vec{k}_2}\left(J_{\vec{k}+\vec{k}_1+\vec{k}_2}+\right.$$

$$\left.+J_{\vec{k}_2}-J_{\vec{k}-\vec{k}_2}-J_{\vec{k}_1-\vec{k}_2}\right)\langle\langle B^+_{\vec{k}_1}B_{\vec{k}_2}B_{\vec{k}+\vec{k}_1-\vec{k}_2}|S^-_{\vec{k}}\rangle\rangle \tag{2.13}$$

$$(E - E^{(0)}_{\vec{k}}) \langle\langle B^+_{\vec{k}_1} B_{\vec{k}_4} B_{\vec{k}+\vec{k}_1-\vec{k}_4} | S^-_{\vec{k}} \rangle\rangle = \frac{i}{2\pi} \Big[\langle B^+_{\vec{k}_1} B_{\vec{k}_1} \rangle_0 (\delta_{\vec{k}_1\vec{k}_4} + \delta_{\vec{k}_4\vec{k}}) -$$

$$- \frac{2}{N} \langle B^+_{\vec{k}_1} B_{\vec{k}_1} \rangle_0 \Big] + \frac{1}{2N} \sum_{\vec{k}_3\vec{k}_4} (J_{\vec{k}+\vec{k}_3-\vec{k}_4} + J_{\vec{k}_4} - J_{\vec{k}-\vec{k}_4} -$$

$$- J_{\vec{k}_3-\vec{k}_4}) \langle\langle B^+_{\vec{k}_1} B_{\vec{k}_2} B_{\vec{k}+\vec{k}_1-\vec{k}_2} | B^+_{\vec{k}+\vec{k}_3-\vec{k}_4} B^+_{\vec{k}_4} B_{\vec{k}_3} \rangle\rangle \qquad (2.14)$$

where

$$\left. \begin{aligned} C &= \frac{1}{N} \sum_{\vec{k}} \langle B^+_{\vec{k}} B_{\vec{k}} \rangle_0 = \frac{1}{N} \sum_{\vec{k}} \left(e^{\frac{E^{(0)}_{\vec{k}}}{kT}} - 1 \right)^{-1} \\ E^{(0)}_{\vec{k}} &= \Delta - \frac{1}{2} J_{\vec{k}} \end{aligned} \right\} \qquad (2.15)$$

and $S^-_{\vec{k}}$ is defined by /2.5/. The mean value in /2.15/ is evaluated using the Hamiltonian /2.7/. It has to be pointed out that in the correlators of Green's functions /2.13/ and /2.14/ all terms proportional to the square and higher orders of C are neglected.

If we combine /2.13/ and /2.14/ we get:

$$(E - E^{(0)}_{\vec{k}}) \cdot \langle\langle S^+_{\vec{k}} | S^-_{\vec{k}} \rangle\rangle = \frac{i}{2\pi} \Big[1 - 2C + \frac{1}{N} \sum_{\vec{q}_4} (J_{\vec{k}} + J_{\vec{q}_4} - J_{\vec{k}-\vec{q}_4} - J_0) \frac{\langle B^+_{\vec{q}_4} B_{\vec{q}_4} \rangle_0}{E - E^{(0)}_{\vec{k}}} \Big] +$$

$$+ \frac{1}{4N^2} \sum_{\vec{k}_1\vec{k}_2\vec{k}_3\vec{k}_4} \frac{1}{E - E^{(0)}_{\vec{k}}} (J_{\vec{k}+\vec{k}_1-\vec{k}_2} + J_{\vec{k}_2} - J_{\vec{k}-\vec{k}_2} - J_{\vec{k}_1-\vec{k}_2}) (J_{\vec{k}+\vec{k}_3-\vec{k}_4} + J_{\vec{k}_4} -$$

$$- J_{\vec{k}-\vec{k}_4} - J_{\vec{k}_3-\vec{k}_4}) \cdot \langle\langle B^+_{\vec{k}_1} B_{\vec{k}_2} B_{\vec{k}+\vec{k}_1-\vec{k}_2} | B^+_{\vec{k}+\vec{k}_3-\vec{k}_4} B^+_{\vec{k}_4} B_{\vec{k}_3} \rangle\rangle \qquad (2.16)$$

Using the Wick's theorem we can decouple Green's functions in /2.16/ in the following way:

$$\langle\langle B_{\vec{k}} | B^+_{\vec{k}+\vec{q}_1-\vec{q}_4} B^+_{\vec{q}_4} B_{\vec{q}_1} \rangle\rangle \simeq \langle B^+_{\vec{q}_1} B_{\vec{q}_1} \rangle_0 \Gamma_{\vec{k}} (\delta_{\vec{q}_4\vec{q}_1} + \delta_{\vec{q}_4\vec{k}}) \qquad (2.17)$$

$$\langle\langle B^+_{\vec{q}_1} B_{\vec{q}_4} B_{\vec{k}+\vec{q}_1-\vec{q}_4} | B^+_{\vec{k}} \rangle\rangle = \langle B^+_{\vec{q}_1} B_{\vec{q}_1} \rangle_0 \Gamma_{\vec{k}} (\delta_{\vec{q}_4\vec{q}_1} + \delta_{\vec{q}_4\vec{k}}) \qquad (2.18)$$

$$\langle\langle B^+_{\vec{q}_1} B_{\vec{q}_2} B_{\vec{k}+\vec{q}_1-\vec{q}_2} | B^+_{\vec{k}+\vec{q}_3-\vec{q}_4} B^+_{\vec{q}_4} B_{\vec{q}_3} \rangle\rangle \simeq \langle B^+_{\vec{q}_1} B_{\vec{q}_1} \rangle_0 \cdot \langle B^+_{\vec{q}_3} B_{\vec{q}_3} \rangle_0 \cdot$$

$$\cdot \Gamma_{\vec{k}} \left(\delta_{\vec{q}_1 \vec{q}_2} \delta_{\vec{q}_3 \vec{q}_4} + \delta_{\vec{q}_1 \vec{q}_2} \delta_{\vec{q}_4 \vec{k}} + \delta_{\vec{q}_2 \vec{k}} \delta_{\vec{q}_3 \vec{q}_4} + \delta_{\vec{q}_2 \vec{k}} \delta_{\vec{q}_4 \vec{k}} \right) \quad (2.19)$$

where

$$\Gamma_{\vec{k}} = \langle\langle B_{\vec{k}} | B^+_{\vec{k}} \rangle\rangle \quad (2.20)$$

If we introduce /2.17/-/2.2o/ into /2.16/ it follows:

$$\left[(E - E^{(0)}_{\vec{k}})(1-2C)^2 - \frac{M^2_{\vec{k}}}{E - E^{(0)}_{\vec{k}}} \right] \Gamma_{\vec{k}} = \frac{i}{2\pi} \left(1 - 2C + \frac{M_{\vec{k}}}{E - E^{(0)}_{\vec{k}}} \right) \quad (2.21)$$

where

$$M_{\vec{k}} = \frac{1}{N} \sum_{\vec{q}} \left(J_{\vec{k}} + J_{\vec{q}} - J_{\vec{k}-\vec{q}} - J_0 \right) \langle B^+_{\vec{q}} B_{\vec{q}} \rangle_0 \quad (2.22)$$

After a simple calculation in which we neglect the square and higher orders of C, the equation /2.21/ reduces to:

$$\Gamma_{\vec{k}} = \frac{i}{2\pi} \frac{1}{E - E^{(1)}_{\vec{k}}} \left(1 + 2C \frac{E - E^{(0)}_{\vec{k}}}{E - E^{(1)}_{\vec{k}}} \right) \quad (2.23)$$

where:

$$E^{(1)}_{\vec{k}} = \Delta - \frac{1}{2} J_{\vec{k}} + \frac{1}{N} \sum_{\vec{q}} \left(J_{\vec{k}} + J_{\vec{q}} - J_{\vec{k}-\vec{q}} - J_0 \right) \langle B^+_{\vec{q}} B_{\vec{q}} \rangle_0 \quad (2.24)$$

We can put $E^{(1)}_{\vec{k}} \simeq E^{(0)}_{\vec{k}}$ in the term $2C \frac{E - E^{(0)}_{\vec{k}}}{E - E^{(1)}_{\vec{k}}} \simeq 2C \left(1 + \frac{M_{\vec{k}}}{E - E^{(1)}_{\vec{k}}} \right)$ of the equation /2.23/ since the difference between $E^{(1)}_{\vec{k}}$ and $E^{(0)}_{\vec{k}}$ would produce in the correlators of Green's functions only the terms proportional to $CM_{\vec{k}} \simeq C^2$ already neglected.

This holds in all complex plane except in the vicinity of a pole where the Green's function has no sence at all. Thus the final expression we get for Green's function is:

$$\Gamma_{\vec{k}} = \frac{i}{2\pi} \frac{1+2C}{E-E^{(1)}_{\vec{k}}} \qquad (2.25)$$

The energy spectrum of elementary excitations is defined as a pole of the boson Green's function /2.25/:

$$E = E^{(1)}_{\vec{k}} \qquad (2.26)$$

and thus we conclude that the elementary excitations with the dispersion law /2.26/ are bosons. The spectral intensity of $\Gamma_{\vec{k}}$:

$$Q(\vec{k},E,kT) = \frac{1+2C}{e^{\frac{E}{kT}}-1}\,\delta(E-E^{(1)}_{\vec{k}}) \qquad (2.27)$$

is necessary in order to find the population number of bosons:

$$\langle B^+_{\vec{n}} B_{\vec{n}} \rangle_Q = \frac{1}{N}\sum_{\vec{k}} \int_{-\infty}^{+\infty} Q(\vec{k},E,kT)\,dE$$

$$= \frac{1}{N}\sum_{\vec{k}}\left(e^{\frac{E^{(1)}_{\vec{k}}}{kT}}-1\right)^{-1} + \frac{2}{N}\sum_{\vec{k}}\left(e^{\frac{E^{(1)}_{\vec{k}}}{kT}}-1\right)^{-1}\cdot\frac{1}{N}\sum_{\vec{k}}\left(e^{\frac{E^{(0)}_{\vec{k}}}{kT}}-1\right)^{-1} \qquad (2.28)$$

which differs from the standard statistical boson formula:

$$\langle B^+_{\vec{n}} B^+_{\vec{n}} \rangle_1 = \frac{1}{N}\sum_{\vec{k}}\left(e^{\frac{E^{(1)}_{\vec{k}}}{kT}}-1\right)^{-1} \qquad (2.29)$$

We find no paradox in this result since, as it is well known from [5], /2.29/ is valid only in a case of a system of noninteracting bosons. So, our conclusion is that boson nature of the elementary excitations is established by /2.25/, but their interactions cause the statistics defined by /2.28/.

3. THE MAGNETIZATION OF A HEISENBERG FERROMAGNET WITH SPIN $S=\frac{1}{2}$

Due to /2.4/ it follows for the magnetization the formula:

$$\sigma_{1/2}=1-2\langle P^{+}_{\vec{m}}P_{\vec{m}}\rangle=1-2\langle B^{+}_{\vec{m}}B_{\vec{m}}\rangle+2\langle B^{+}_{\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}}B_{\vec{m}}\rangle \qquad (3.1)$$

By the use of the Wick's theoreme the last term in /3.1/ can be transformed in the following way

$$\langle B^{+}_{\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}}B_{\vec{m}}\rangle_{0}=2\left(\langle B^{+}_{\vec{m}}B_{\vec{m}}\rangle_{0}\right) \qquad (3.2)$$

So we have:

$$\sigma_{1/2}=1-2\langle B^{+}_{\vec{m}}B_{\vec{m}}\rangle_{0}+4\left(\langle B_{\vec{m}}B_{\vec{m}}\rangle_{0}\right)^{2} \qquad (3.3)$$

After the averaging and the neglection of the terms proportional to C^{3} one obtains:

$$\sigma_{1/2}=1-\frac{2}{N}\sum_{\vec{k}}\left(e^{\frac{E^{(0)}_{\vec{k}}}{kT}}-1\right)^{-1} \qquad (3.4)$$

By the usual procedure it follows from /2.26/ and /3.4/ the final expression for the magnetization:

$$\left.\begin{aligned}&\sigma_{1/2}=1-2Z_{3/2}(\alpha)\tau^{3/2}-\frac{3\pi}{2}Z_{5/2}(\alpha)\tau^{5/2}-\frac{33\pi^{2}}{16}Z_{7/2}(\alpha)\tau^{7/2}-6\pi Z_{3/2}(\alpha)\tau^{4}+O(\tau^{9/2})\\&Z_{p}(\alpha)=\sum_{n=1}^{\infty}n^{-p}e^{-n\alpha}\,;\ \alpha=\frac{\mu\mathcal{H}}{kT}\,;\ \tau=\frac{kT}{2\pi I}\end{aligned}\right\}\qquad (3.5)$$

identical to the result of Dyson.

As for the approximations we used in the calculations of Section 2. and 3., we can say that the terms proportional to C^2 were neglected since these would give the corrections of the order C^3 what is evident from the formula /2.28/ for

If we add in the Hamiltonian the sixth-order terms describing the three-particle processes /i.e. the terms of the type $B^{+3}B^3$/, the resulting Green's function would be of the type $\langle\langle B^{+2}B^3/B^+\rangle\rangle$ i.e. the leading term in its correlator would be proportional to C^2. The mathematical proof of this statement is elementary but rather tedious.

4. CONCLUSION

In conclusion we emphasize the following:

a/ The method we developed in this paper suffers neither from any inconsistencies inherent to the previous approaches nor it is mathematically complicated.

b/ As a final consequence of this new approach the completely new physical picture of the elementary excitations in the ferromagnet arises according to which the elementary excitations really existing in the ferromagnet are bosons interacting between them in such a way to increase the population compared to the system of noninteracting bosons. This is quite contrary to the opinions usually accepted that the magnons are not Bose particles but have the statistics of a system of noninteracting bosons.

Also in this paper the following methodological innovations are introduced:

c/ The low temperature analysis of the ferromagnet was carried out in terms of the exact boson representation of spin operators.

d/ We started from the Hamiltonian and spin operators written in momentum space and used the decoupling procedure /2.17/-/2.19/

more general than the usual one [4].

c/ The results of Dyson are conffirmed but in the qualitetively new picture explained above.

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ABSTRACT

The energy spectrum and the magnetization of Heisenberg ferromagnet with spin $S=\frac{1}{2}$ are evaluated in terms of the exact boson representation of the spin $S=\frac{1}{2}$. In order to get the Green's function of the system the nonstandard treatment was used in which Heisenberg equations of motion are immediatelly set in momentum space. As a final consequence of this approach the new physical picture of the ferromagnet arises according to which the real elementary excitations are bosons but with the statistical formula different from the standard one.

THE LOW TEMPERATURE THEORY OF BOSE CONDENSATION IN HEISENBERG FERROMAGNET WITH SPIN $S=\frac{1}{2}$

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1. INTRODUCTION

The possibility of Bose condensation in a quasiparticle system has been investigated in a number of papers [1-6] published during the last ten years. The main conclusion of these papers is that Bose condensation is probable if the time t_c for the system to achieve the state of thermodynamical equilibrium (due to the quasiparticle collisions) is shorter than the quasiparticle life-time t_ℓ . Since $t_c=a^{-2}n^{-1}v^{-1}$ where $a\sim10^{-8}cm$ is the magnon's size, $n\sim10^{18}-10^{20}$ -the magnon concentration in crystal per cm^3 (this follows from the Bloch's „$T^{\frac{3}{2}}$" law at low temperatures) and $v\sim10^{6}\frac{cm}{s}$ -the magnon velocity, it follows that the value of t_c is of the order $10^{-8}-10^{-10}s$. On the other hand the experiments [7] give for the magnon life-time the value $t_\ell\sim10^{-5}s$ (for the strong external magnetic fields), i.e. at the low temperatures the condition for the existence of the condensat is fulfilled.

The Bose condensation will be considered in this paper as a statistical fluctuation occuring in the time interval $10^{-8}s<t<10^{-5}s$ Without the relevant experimental facts it seems to us that nothing definitively can be said for the case of the stable thermodynamical states. The experimental investigations of Bose condensation considered as a statistical fluctuation can be examined in the crystal exposed to the neutron beam of high intensity. The measurements have to be performed just after (in the interval of time smaller than $10^{-8}s$) the transition of neutrons through the crystal.

The primary aim of this paper is to investigate the energy spectrum of elementary excitations in the ferromagnet in which Bose condensation occurs. Therefore the possible processes in the ferromagnet are analysed in the second section where we shall demonstrate that the scattering of quasiparticles is to take place dominantly in the case of the strong external magnetic fields while in the ferromagnet in the weak magnetic fields the binding of two bosons into the new localized quasiparticle appears to be the concurent process to that of the scattering.

The energy spectrum of elementary excitations in the Heisenberg ferromagnet is investigated in details in the third part of this paper using the Green's functions method. The main result of this section is that (if the intensity of the wave vector $\vec{k}$ is small compared to the square root of the concentration n_0 of the condensed bosons) the dispersion law depends linearly on $|\vec{k}|$ while for $|\vec{k}|\gg\sqrt{n_0}$ we obtain the dispersion law depending on $|\vec{k}|^2$

The possible processes in the ferromagnet under the weak magnetic field are discussed in the fourth section. Since the contributions coming from the processes of binding of two bosons at one lattice point are exponentially small at the low temperatures, the dispersion law and the magnetization of the ferromagnet appeared to be given by the well-known Dyson's formulae.

2. ON THE PROCESSES AT δ-POTENTIAL

The Pauli representation of the Hamiltonian of the Heisenberg ferromagnet with spin $S=\frac{1}{2}$ and simple cubic lattice has the following form

$$H=H_0+\Delta\sum_{\vec{m}} P^+_{\vec{m}} P_{\vec{m}} - \frac{1}{2}\sum_{\vec{m},\vec{n}} I_{\vec{m}\vec{n}} P^+_{\vec{m}} P_{\vec{n}} - \frac{1}{2}\sum_{\vec{m}\vec{n}} P^+_{\vec{m}} P^+_{\vec{n}} P_{\vec{n}} P_{\vec{m}} I_{\vec{m}\vec{n}} \quad (2.1)$$

where

$$H_0=-\frac{1}{2}N\mu\mathcal{H}-\frac{1}{8}NJ_0 \quad (2.2)$$

N -the number of atoms in the crystal,

μ -the magnetic moment of an atom,

$\mathcal{H}$ -the external magnetic field,

$$J_0=\sum_{\vec{m}} I_{0\vec{m}}$$

$I_{\vec{m}\vec{n}}$ -the exchange integral

$$\Delta=\mu\mathcal{H}+\frac{1}{2}J_0$$

$P^+_{\vec{m}}$ and $P_{\vec{m}}$ -the Pauli operators satisfying the following commutation relations:

$$\left.\begin{array}{c} \left[P_{\vec{m}}, P^+_{\vec{n}}\right]=(1-2P^+_{\vec{m}}P_{\vec{m}})\delta_{\vec{m}\vec{n}} \\ (P_{\vec{m}})^2=(P^+_{\vec{m}})^2=0 \\ \left[P^+_{\vec{m}}, P^+_{\vec{n}}\right]=\left[P_{\vec{m}}, P_{\vec{n}}\right]=0 \\ L_{\vec{m}}=0 \text{ or } L_{\vec{m}}=1 \end{array}\right\} \quad (2.3)$$

where $L_{\vec{m}}$ is the eigen-value of the operator $P^+_{\vec{m}}P_{\vec{m}}$.

Since the Fourier transformation of the Pauli operators $P^{+}_{\vec{m}}$ and $P_{\vec{m}}$ does not conserve the commutation relations (2.3) we shall use the exact boson representation of the Pauli operators [4]. At the low temperatures it is enough to take into account the two-boson processes only:

$$P_{\vec{m}}=B_{\vec{m}}-B^{+}_{\vec{m}}B_{\vec{m}}B_{\vec{m}}\,;\; P^{+}_{\vec{m}}=B^{+}_{\vec{m}}-B^{+}_{\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}}\,;\; P^{+}_{\vec{m}}P_{\vec{m}}=B^{+}_{\vec{m}}B_{\vec{m}}-(B^{+}_{\vec{m}})^{2}(B_{\vec{m}})^{2} \quad (2.4)$$

Using (2.1) and (2.5) we obtain the boson Hamiltonian of the system:

$$H=H_{ASQ}+H_{\delta}+H_{c_1}+H_{c_2}+H_{D} \quad (2.5)$$

where:

$$H_{ASQ}=\Delta\sum_{\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}}-\frac{1}{2}\sum_{\vec{m}\vec{m}}I_{\vec{m}\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}} \quad (2.6)$$

$$H_{\delta}=-\frac{\Delta}{2}\sum_{\vec{m}\vec{m}}2\delta_{\vec{m}\vec{m}}B^{+}_{\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}}B_{\vec{m}} \quad (2.7)$$

$$H_{c_2}=\frac{1}{2}\sum_{\vec{m}\vec{m}}I_{\vec{m}\vec{m}}B^{+}_{\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}}B_{\vec{m}} \quad (2.8a)$$

$$H_{c_1}=\frac{1}{2}\sum_{\vec{m}\vec{m}}I_{\vec{m}\vec{m}}B^{+}_{\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}}B_{\vec{m}} \quad (2.8b)$$

$$H_{D}=-\frac{1}{2}\sum_{\vec{m}\vec{m}}I_{\vec{m}\vec{m}}B^{+}_{\vec{m}}B^{+}_{\vec{m}}B_{\vec{m}}B_{\vec{m}} \quad (2.9)$$

H_{c_1}, H_{c_2} and H_{δ} have to be accounted for the kinematical interaction of elementary excitations while H_D corresponds to the dynamical interaction. In further calculations with H_{c_1}, H_{c_2} and H_D we shall use the Born approximation because these terms are composed of the potentials of the Born's type.

Starting from the general theory of I.M.Lifšic [8], Konobeev and Dubovskij [9] deduced the following formula for the δ -potential scattering length:

$$f(|\vec{k}|,\Delta)=\frac{a}{2}\frac{\xi}{1-\xi+\xi y^2-i\frac{\pi}{2}\xi y} \tag{2.10}$$

where:

a -the lattice constant,

$$\xi=\frac{\Delta}{\pi I} \tag{2.11}$$

I -the exchange integral for the nearest neighbours,

$$y=\frac{|\vec{k}|}{|\vec{k}_m|}$$

$\vec{k}_m$ -the maximal value of the wave vector in the first Brillouin zone.

It has to be pointed out that (2.10) represents the corresponding formula from the paper [9] adapted for the ferromagnet where we take as the reduced mass of two magnons the value

$$m=\frac{1}{2Ia^2} \tag{2.12}$$

At the low temperatures only the excitations having the small values of $|\vec{k}|$ exist in the ferromagnet, so (2.10) can be reduced to:

$$f=\frac{a}{2}\frac{\xi}{1-\xi} \tag{2.13}$$

Therefore for the strong external magnetic fields $(\mu\mathcal{H}\gg I)$ we have for the scattering length from (2.13) the following value:

$$f=-\frac{a}{2} \tag{2.14}$$

Thus we conclude that there is the scattering of quasiparticles at δ -potential in this case.

If we do the standard transition in H_δ (see,for example[10]

$$\sum_{\vec{n}\vec{m}} U_{\vec{n}\vec{m}} B^+_{\vec{n}} B^+_{\vec{m}} B_{\vec{n}} B_{\vec{m}} \rightarrow \frac{4\pi f}{ma^3} \sum_{\vec{n}} B^+_{\vec{n}} B^+_{\vec{n}} B_{\vec{n}} B_{\vec{n}} \qquad (2.15)$$

it follows from (2.7), (2.12), (2.14) and (2.15):

$$H_\delta = 2\pi I \sum_{\vec{n}} B^+_{\vec{n}} B^+_{\vec{n}} B_{\vec{n}} B_{\vec{n}} \qquad (2.16)$$

i.e. H_δ describes the repulsion of bosons.

For the small external fields when $\mu \mathcal{H} \ll I$ (this includes also $\mathcal{H}=0$) the formula (2.13) gives:

$$f = \frac{a}{2} \frac{3}{\pi - 3} \qquad (2.17)$$

i.e. the smallest distance of two quasiparticles is much larger than the quasiparticle size.In fact,this means that there was no scattering at one lattice point (because in the cases when the scattering takes place the scattering lenghth is of the order of the quasiparticle size)but the binding of two bosons into the new quasiprticle having the size much larger than the size of the initial quasiprticles i.e. 10^{-8} cm .Since this quasiparticle has the large size we can conclude that it is localized (see also [8]) and therefore it's dispersion law does not depend on $\vec{k}$ practically.The corresponding population number of these quasiparticles is proportional to $e^{-\frac{Const}{kT}}$,so the contributions coming from these quasiparticles are very small at the low temperatures what is the well-known result of Dyson. It has to mentioned that H_{c_1} also leads to the binding of two bosons at one lattice point while H_{c_2}

and H_D describe the scattering of two bosons.

To conclude this section we emphasize the following: in strong external magnetic fields at low temperatures and in short time intervals only the scattering of bosons occurs in the ferromagnet, while in the case of weak fields, besides scattering, there is the binding of two bosons at one lattice point into a new, strogly localized quasiparticle.

2. THE ENERGY SPECTRUM OF THE FERROMAGNET UNDER THE STRONG EXTERNAL MAGNETIC FIELD

The method of analysis of the elementary excitations energy spectrum in the quasiparticle system in which the condensat is formed has been developed by Beljaev [11]. We shall apply here the Belyaev's tehnique of Green's functions in order to find the energy spectrum of elementary excitations in the ferromagnet when in it the Bose condensation takes place.

Bose operator $B_{\vec{n}}$ can be decomposed in a following way:

$$B_{\vec{n}} = b_{o\vec{n}} + b_{\vec{n}} \tag{3.1}$$

where $b_{o\vec{n}}$ corresponds to the bosons which are in the condensat and have the zero-momentum while $b_{\vec{n}}$ corresponds to bosons having non-zero momentum. The former bosons will be called the condensed ones and the and the latter the subcondensed ones. The plane wave expansion of the operator $B_{\vec{n}}$

$$B_{\vec{n}} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} B_{\vec{k}} e^{i\vec{k}\vec{n}} = \frac{B_o}{\sqrt{N}} + \frac{1}{\sqrt{N}} \sum_{\vec{k}\neq 0} B_{\vec{k}} e^{i\vec{k}\vec{n}} \tag{3.2}$$

gives for $b_{o\vec{n}}$ and $b_{\vec{n}}$ the following:

$$b_{o\vec{n}} = \frac{B_o}{\sqrt{N}} \; ; \quad b_{\vec{n}} = \frac{1}{\sqrt{N}} \sum_{\vec{k}\neq 0} B_{\vec{k}} e^{i\vec{k}\vec{n}} \tag{3.3}$$

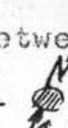
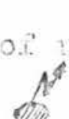
Bogoliubov has shown that the operators $b_{o\vec{n}}$ slightly differ from numbers, i.e.

$$\left[b_{0\vec{k}}, b_{0\vec{k}}^{+} \right] \sim \frac{1}{N} \qquad (3.4)$$

Therefore the eigenvalues of the operators $b_{0\vec{k}}$ and $b_{0\vec{k}}^{+}$ are:

$$b_{0\vec{k}} = b_{0\vec{k}}^{+} = \sqrt{\frac{N_0}{N}} \equiv \sqrt{n_0} \qquad (3.5)$$

where N_0 is the number of bosons in the condensed state. The separation of Bose operators $B_{\vec{k}}$ into a pair of Bose operators simply reflects the fact we actually have the mixture of two interacting subsystems.

The following processes have to be taken into account when one calculates the energy spectrum of elementary excitations in a system with the condensed state: the scattering of noncondensed bosons, the annihilation of two condensed bosons with the simultaneous creation of two noncondensed bosons and the inverse process also. These processes will be described by the Green's functions $G^{N,N}(\vec{k})$, $G^{N,N+2}(\vec{k})$ and $G^{N,N-2}(\vec{k})$, respectively. The complete description of the system with the condensat is given by the set of equations defining $G^{N,N}(\vec{k})$ and $G^{N,N+2}(\vec{k})$ only, which is represented on Fig.1. The symbol  denotes the Green's function of noncondensed bosons,  represents the Green's function of free bosons (noncondensed also) and  is the Green's function appearing due to interactions between noncondensed and condensed bosons. The vertex $M_{11}(\vec{k}) =$  corresponds to the scattering of the noncondensed bosons and to the scattering of noncondensed bosons at the condensed ones. The vertex $M_{20}(\vec{k}) =$  describes the creation of two noncondensed bosons with the simultaneous annihilation of two condensed ones, and $M_{02}(\vec{k}) =$  is to be accounted for the inverse process. The value $\sqrt{n_0}$ has to be joined to the each line N . The Green's function ⟶ is given in the case of ferromagnet by:

$$G_0^{N,N}(\vec{k}) = \frac{1}{\omega - \frac{1}{2} I |\vec{k}|^2 + \lambda_0} \qquad (3.6)$$

where

$$\left.\begin{aligned} \omega &= E - \mu \mathcal{H} \\ \lambda_0 &= M_{11}(0) - M_{20}(0) \end{aligned}\right\} \qquad (3.7)$$

The analytical expression of the equations from Fig.1. is the following:

$$\left.\begin{aligned} G^{N,N}(\vec{k}) &= \frac{\omega + \frac{1}{2} I |\vec{k}|^2 + S(\vec{k}) + A(\vec{k}) - \lambda_0}{[\omega - A(\vec{k})]^2 - [\frac{1}{2} I |\vec{k}|^2 + S(\vec{k}) - \lambda_0]^2 + M_{20}(\vec{k}) M_{02}(\vec{k})} \\ G^{N,N+2}(\vec{k}) &= \frac{M_{02}(\vec{k})}{[\omega - A(\vec{k})]^2 + [\frac{1}{2} I |\vec{k}|^2 + S(\vec{k}) - \lambda_0]^2 + M_{20}(\vec{k}) M_{02}(\vec{k})} \end{aligned}\right\} \qquad (3.8)$$

where

$$\left.\begin{aligned} S(\vec{k}) &= \frac{1}{2}\left[M_{11}(\vec{k}) + M_{11}(-\vec{k})\right] \\ A(\vec{k}) &= \frac{1}{2}\left[M_{11}(\vec{k}) - M_{11}(-\vec{k})\right] \end{aligned}\right\} \qquad (3.9)$$

The energy $E(\vec{k})$ of elementary excitations (defined as a pole of the both Green's functions) is given as a solution of the following equation:

$$\left[\omega - A(\vec{k})\right]^2 - \left[\frac{1}{2}I|\vec{k}|^2 + S(\vec{k}) - \lambda_0\right]^2 + M_{20}(\vec{k}) M_{02}(\vec{k}) = 0 \quad (3.10)$$

It is necessary now to calculate the vertex parts $M_{11}(\vec{k})$, $M_{20}(\vec{k})$ and $M_{02}(\vec{k})$. The graphs describing the processes in the system are given on Fig.2 and Fig.3. It must be pointed out that the graphs describe the first order processes only. The black point on the Fig.2. denotes δ - potential while the wave line corresponds to the potential $I_{\vec{m}\vec{m}'}$. The graphs containing condensed boson lines only are not explicitely given on Fig.2. but their contributions are taken into account on the Fig.2. by help of the Hugenholc- ines theoreme (see [10], ch.V). According to this theoreme every graph containing condensed lines only is equal to the sum of all possible graphs obtained from it by the substitution of pairs of condensed lines by the pairs of the noncondensed ones. The graphs (A_1), (A_6), (A_{11}) and (A_{16}) which contain the full lines only corresponding to noncondensed bosons, are evaluated by a standard technique on Fig.3. in the first order perturbation. The broken lines on Fig.3. correspond to the mean value of the number of noncondensed bosons and, consequently, they should be replaced by

$$\bar{n} = \frac{1}{N} \sum_{\vec{q} \neq 0} \langle b^{+}_{\vec{q}} b_{\vec{q}} \rangle \equiv \frac{1}{N} \sum_{\vec{q} \neq 0} n_{\vec{q}} \quad (3.11)$$

As it was pointed out [10] the higher order graphs give the contributions proportional either to n_0^2 or to $n_0 \bar{n}$, and due to the small concentrations n_0 and $\bar{n}$ these contributions may be neglected.

According to Figs. 2. and 3. one obtains the following expressions for the vertex parts:

$$\left.\begin{aligned} M_{11}(\vec{k}) &= n_0(8\pi I + J_0 + J_{\vec{k}}) + \frac{1}{N}\sum_{\vec{q}\neq 0} n_{\vec{q}}(8\pi I + 2J_{\vec{k}} - J_0 - J_{\vec{k}-\vec{q}}) \\ M_{20}(\vec{k}) &= M_{02}(\vec{k}) = n_0(4\pi I + J_0) \end{aligned}\right\} \quad (3.12)$$

Using (3.11) and (3.12) we have evaluated λ_0 from (3.7) and $S(\vec{k})$ and $A(\vec{k})$ from (3.9) .The energy of elementary excitations in the ferromagnet with Bose condensat as it follows from is:

$$E(\vec{k}) = \mu H + \sqrt{\left[\frac{1}{2}I|\vec{k}|^2 + \alpha(\vec{k})\right]^2 - \beta^2} \quad (3.13)$$

where:

$$\left.\begin{aligned} \alpha(\vec{k}) &= (2\pi I + J_{\vec{k}})n_0 + \frac{1}{N}\sum_{\vec{q}\neq 0} n_{\vec{q}}\left[2(J_{\vec{k}} - J_0) + J_{\vec{q}} - J_{\vec{k}-\vec{q}}\right] \\ \beta &= (2\pi I + J_0)n_0 \end{aligned}\right\} \quad (3.14)$$

The boson spectrum under the conditions of condensation satisfies the Goldstone theorem because when $|\vec{k}| \to 0$ then the energy $E(\vec{k}) \to \mu H$ .The energy $E(\vec{k})$ depends linearly on $|\vec{k}|$ when $|\vec{k}| \ll \sqrt{n_0}$,i.e. quasiparticles have the "acoustic" dispersion law.The velocity of this "magnon sound" is given by:

$$E(\vec{k}) - \mu H \simeq \sqrt{I\beta}\,|\vec{k}| \quad (3.15)$$

On the other hand,when $|\vec{k}| \gg \sqrt{n_0}$,it follows from (3.13) the quadratic dispersion law $E(\vec{k}) \approx const + \frac{1}{2}I|\vec{k}|^2$,i.e. such dispersion law which we have in the cases when the condensat does not exist.

4. THE ENERGY SPECTRUM OF THE FERROMAGNET UNDER THE WEAK MAGNETIC FIELD

Two processes take place simutaneously with the equal velocities in the ferromagnet in the weak magnetic fields:the scattering of free bosons and the binding of two bosons when they appear at the same lattice point.The time t_c in which the ferromagnet achieves the state of the thermodynamical equilibrium and life-time t_ℓ of free bosons are of the same order of magnitude and therefore the appearance of the condensed state does not seem to be sure,as in the case of the strong external magnetic fields.We shall assume, however,that there is the condensat in the ferromagnet,but all the graphs containing two incoming vertex lines (those are the graphs $(A_1)-(A_{10})$ on Fig.2) which correspond to the appearance of two bosons at the same lattice point will be rejected since these graphs give exponentially small corrections to the thermodynamical characteristics of the system,as it was pointed out in Section2. The evaluation of the vertex parts using the graphs $(A_{11})-(A_{20})$ from Fig.2. (and corresponding graphs from Fig.3.) gives:

$$\left.\begin{aligned} &\tilde{M}_{11}(\vec{k}) = \frac{1}{N}\sum_{\vec{q}\neq 0} n_{\vec{q}}\left(J_{\vec{k}} + J_{\vec{q}} - J_0 - J_{\vec{k}-\vec{q}}\right) \\ &\tilde{M}_{20}(\vec{k}) = 0 \;;\quad \tilde{M}_{02}(\vec{k}) = n_0\left(J_0 - J_{\vec{k}}\right) \end{aligned}\right\} \qquad (4.1)$$

and.

$$\tilde{S}(\vec{k}) = \tilde{M}_{11}(\vec{k})\;;\; \tilde{A}(\vec{k}) = 0\;;\; \tilde{\lambda}_0 = \tilde{M}_{11}(0) - \tilde{M}_{20}(0) = 0 \qquad (4.2)$$

The energy spectrum, obtained from (3.10),

$$\tilde{E}(\vec{k}) = \frac{1}{2} I |\vec{k}|^2 + \mu \mathcal{H} + \frac{1}{N} \sum_{\vec{q} \neq 0} n_{\vec{q}} \left(J_{\vec{k}} + J_{\vec{q}} - J_0 - J_{\vec{k}-\vec{q}} \right) \quad (4.3)$$

is identical to the well-known dispersion law of Dyson[12]

The magnetization is given by

$$\sigma = 1 - 2 \langle P^{+}_{\vec{m}} P_{\vec{m}} \rangle \quad (4.4)$$

or using the exact Bose representation of Pauli operators (2.5), by

$$\sigma = 1 - 2 \langle B^{+}_{\vec{m}} B_{\vec{m}} \rangle + 2 \langle B^{+}_{\vec{m}} B^{+}_{\vec{m}} B_{\vec{m}} B_{\vec{m}} \rangle \quad (4.5)$$

The last term in (4.5) has to be rejected since it, as the graphs (A$_1$)-(A$_5$), corresponds to the appearance of two bosons at one lattic point and, consequently, gives the exponentially small corrections. If we substitute in

$$\sigma = 1 - 2 \langle B^{+}_{\vec{m}} B_{\vec{m}} \rangle \quad (4.6)$$

the mean value

$$\langle B^{+}_{\vec{m}} B_{\vec{m}} \rangle = \frac{1}{N} \sum_{\vec{q}} \left(e^{\frac{E(\vec{k})}{kT}} - 1 \right)^{-1} \quad (4.7)$$

it follows the Dyson's formula for magnetization;

$$\left. \begin{aligned} \sigma &= 1 - 2 Z_{3/2}(\alpha) \tau^{3/2} - \frac{3\pi}{2} Z_{5/2}(\alpha) \tau^{5/2} - \frac{33\pi^2}{16} Z_{7/2}(\alpha) \tau^{7/2} - 6\pi Z_{3/2}(\alpha) \tau^{4} + O(\tau^{9/2}) \\ Z_p(\alpha) &= \sum_{n=1}^{\infty} n^{-p} e^{-n\alpha} \; ; \; \alpha = \frac{\mu \mathcal{H}}{kT} \; ; \; \tau = \frac{kT}{2\pi I} \end{aligned} \right\} \quad (4.8)$$

So we can conclude that all the results of Dyson's theory of ferro magnet are valid independently on the presence of the condensat if the ferromagnet is in the weak external magnetic field.

5. CONCLUSION

The possible processes in the ferromagnet with spin $S=\frac{1}{2}$ are in details investigated in this paper. It was found that Bose condensation is probable if ferromagnet is in the strong external magnetic field. Considering Bose condensation as a statistical fluctuation of a life-time approximately 10^{-5}s it is established the energy spectrum of elementary excitations depending linearly on the wave vector intensity $|\vec{k}|$ if it is much smaller than the square root of concentration of the condensed bosons - $\sqrt{n_0}$, while for $|\vec{k}| \gg \sqrt{n_0}$ it is found the quadratic dispersion law.

It was shown also that, besides Bose condensation, the binding of two bosons into a new, localized quasiparticle takes place in the ferromagnet if it is placed into a weak external magnetic field. It turned out, however, that in this case the Dyson's results hold regardless of the presence of the condensed state.

In the case of strong magnetic fields, in time-intervals much longer than 10^{-5}s the binding of two bosons into localized quasiparticle is possible also, so that, by the same kind of reasoning as in Section 4. we can obtain the Dyson's results in this case also. Naturally, only the experiment can give the answer what really happens in the ferromagnet under the strong magnetic fields and in long long time-intervals, i.e. which process is dominant : scattering of free bosons or their binding into a localized state.

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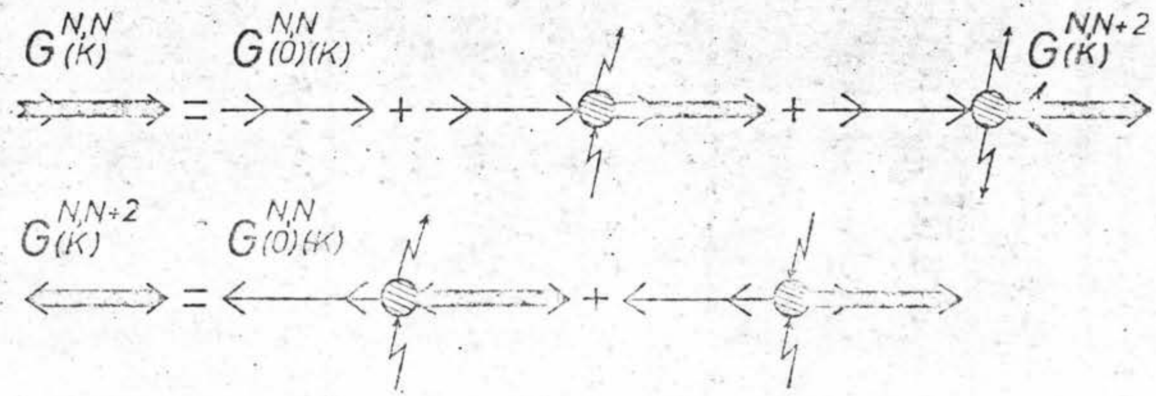

Fig.1

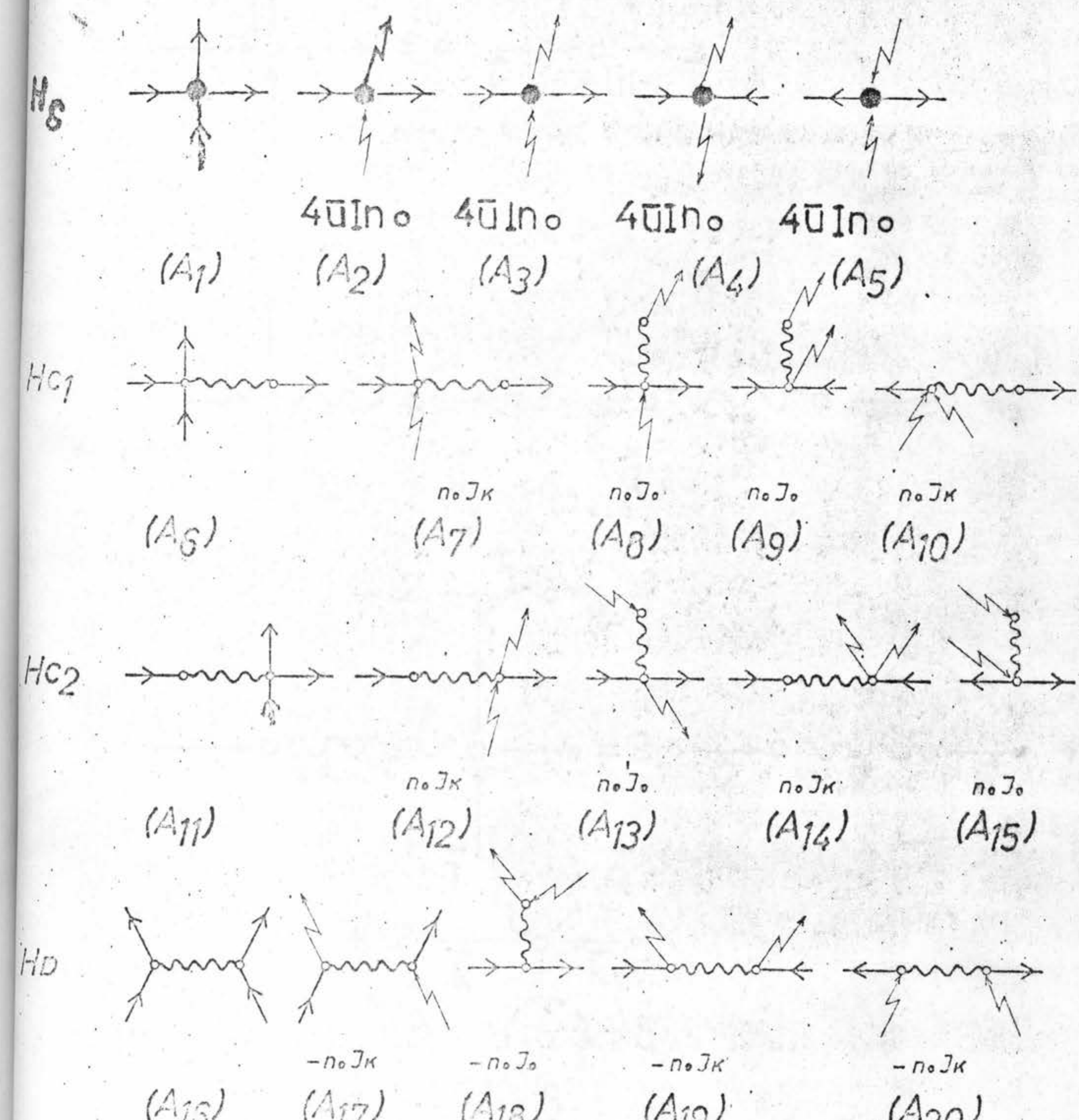
H_S
4ūIn₀ 4ūIn₀ 4ūIn₀ 4ūIn₀
(A1) (A2) (A3) (A4) (A5)
Hc1
n₀Jκ n₀J₀ n₀J₀ n₀Jκ
(A6) (A7) (A8) (A9) (A10)
Hc2
n₀Jκ n₀J₀ n₀Jκ n₀J₀
(A11) (A12) (A13) (A14) (A15)
HD
−n₀Jκ −n₀J₀ −n₀Jκ −n₀Jκ
(A16) (A17) (A18) (A19) (A20)

Fig. 2

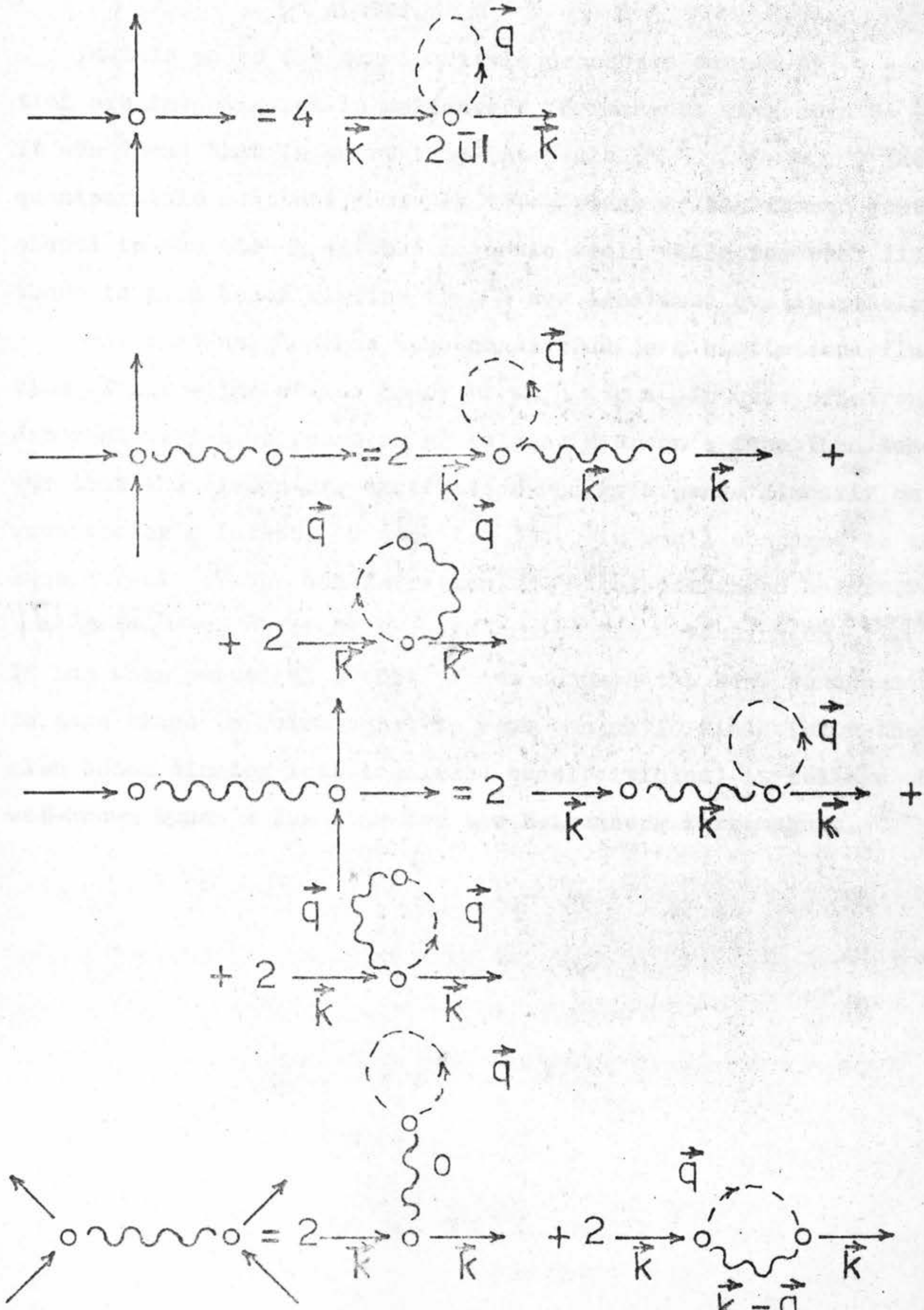

Fig. 3

ABSTRACT

In this paper the quasiparticle processes caused by δ -potential are investigated in Heisenberg ferromagnet with spin $S=\frac{1}{2}$ It was found that in short time-intervals ($t<10^{-5}s$) the quasiparticle scattering mostly takes place if the ferromagnet is placed in the strong external magnetic field while for weak fields there is also boson binding into a new localized quasiparticle.

Bose condensation has been considered as a statistical fluctuation of life-time of the order $10^{-8}s<t<10^{-5}s$ .In case of strong external fields we found, using Belyaev's Green's functions tehnique that the elementary excitations energy depends linearly on wave vector's intensity $|\vec{k}|$ if $|\vec{k}|$ is small compared to the square root of the concentration n_0 of condensed bosons. For $|\vec{k}|\gg\sqrt{n_0}$ we obtained the dispersion law depending on $|\vec{k}|^2$. It has been proved also that if one assumes the Bose condensation to take place in ferromagnet in weak magnetic field (when there is also boson binding into localized quasiparticles) it follows the well-known Dyson's formulae for the Heisenberg ferromagnet.

Glava druga

Vezana stanja u Heisenberg-ovom feromagnetu.

A. Razlozi, ciljevi i metodi

Problem vezanih stanja u Heisenberg-ovom feromagnetu predstavlja jedan od retkih problema u teoriji feromagnetizma koji se može egzaktno rešiti. Upravo iz tog razloga on je često odabiran kao test za valjanost raznih formalizama. U izučavanju ovog problema, međutim, ipak se nailazilo na teškoće koje će ovde biti izložene da bi se razumelo zbog čega je rad na ovom problemu postao jedan od predmeta ove disertacije.

Kao što je dobro poznato dvomagnonska stanja u Heisenberg-ovom feromagnetu definišu se preko talasne funkcije.

$$\Psi = \sum_{\vec{i}\vec{j}} A_{\vec{i}\vec{j}} S^-_{\vec{i}} S^-_{\vec{j}} |0\rangle \qquad /2.1/$$

gde $A_{\vec{i}\vec{j}}$ predstavlja amplitudu verovatnoće da spinski operatori $S^-_{\vec{i}}$ i $S^-_{\vec{j}}$ stvore, delujući na osnovno stanje $\Psi_0 = |0\rangle$ /u kojem su svi spinovi "gore"/ spinove "dole" na mestima $\vec{i}$ i $\vec{j}$ u kristalnoj rešetci. Odmah primećujemo da za spin $S=\frac{1}{2}$ ova funkcija nema nikakvog smisla jer je čitav skup stanja $S^-_{\vec{i}} S^-_{\vec{i}} |0\rangle = 0$ pa odgovarajući koeficijenti $A_{\vec{i}\vec{i}}$ mogu da imaju proizvoljne vrednosti. I pored toga ova funkcija je bila korišćena, od strane niza autora, u proceduri kojom se dobijaju vrednosti energije za spin $S=\frac{1}{2}$. Pritom je, naravno, budući da se svesno polazilo od nečega što nema smisla, bilo potrebno da se kasnije, pošto se formira jednačina koja opisuje dvomagnonska stanja, učine ispravke. Bethe[18], na primer, postavlja dopunski uslov $A_{ii} + A_{jj} = A_{ij} + A_{ji}$ /i,j najbliži susedi u jednodimenzionalnoj rešetci/, dok Mattis[20] smatra da treba dopustiti definisanje nefizičkih amplituda $A_{\vec{i}\vec{i}}$, i da ceo postupak određivanja energije i talasne funkcije dvomagnonskog stanja u slučaju spina $S=\frac{1}{2}$ treba sprovesti kao za spin $S > \frac{1}{2}$ kada nema pomenute teškoće.

Glavna mana a ujedno i vrlina, ovih "rešenja" jeste njihova heurističnost. Dok Bethe ad hoc postavlja svoj uslov, Mattis prelazi preko ove ozbiljne teškoće jer u protivnom jednačina koja opisuje dvomagnonska stanja ne bi mogla ni da se formira.

Ideja kojom ćemo se mi rukovoditi u rešavanju ove teškoće jeste da, pošto već ima ozbiljnih indikacija da su elementarne ekscitacije u feromagnetu bozonskog karaktera[10, 14], kao funkciju pobuđenog stanja treba uzeti

$$\Psi = \sum_{\vec{i}\vec{j}} A_{\vec{i}\vec{j}} B^{+}_{\vec{i}} B^{+}_{\vec{j}} \qquad /2.2/$$

gde operatori $B^{+}_{\vec{i}}$ $B^{+}_{\vec{j}}$ stvaraju Bose "čestice" na mestima $\vec{i}$ i $\vec{j}$. Zatim treba, spinske operatore u Hamiltonijanu izraziti pomoću egzaktne bozonske reprezentacije spinskih operatora[8),14] i naći energiju i koeficijente $A_{\vec{i}\vec{j}}$ u talasnoj funkciji stanja, a na kraju potražiti procese u feromagnetu u kojima se ova razlika u prilazima jasno očituje za fizičke rezultate. Na ovaj način bi pored rešenja pomenute teškoće bio eksplicitno dat metod koji, u principu, daje odgovor na pitanje da li su elementarne ekscitacije u feromagnetu spinskog ili bozonskog karaktera.

Ujedno, ovakav novi prilaz rešava drugu teškoću koja se javlja pri analizi pomoću spinskih operatora, a koja se sastoji u nekanoničnosti njihovih komutacionih relacija prilikom Fourier-ove transformacije kojom se prelazi iz prostora direktne u prostor recipročne rešetke. Nekanoničnost ove transformacije unosi, kao što je već naglašeno u prvoj glavi, mogućnost pravljenja greški koje je nemoguće kontrolisati. Međutim, kao što je poznato, Fourier-ova transformacija bozonskih operatora održava njihove komutacione relacije pa pri radu sa bozonskim operatorima ovaj problem ne postoji.

U analizi problema vezanih stanja koristićemo pored egzaktne bozonske reprezentacije spinskih operatora, i druge bozonske reprezentacije /Dyson-ovu[4], Marumori-jevu[20] da bi detaljno ispitali koja reprezentacija najadekvatnije opisuje vezana stanja i procese u feromagnetu u koje su ona uključena.

B. Rezultati

U radovima 21), 22), 23) priloženim u ovoj glavi dobijeni su sledeći rezultati:

1. Energije vezanih stanja dva bozona i dva magnona u Heisenberg-ovom feromagnetu sa spinom $S \neq \frac{1}{2}$ su identične[23] /formula 2.1.7/ i definisane sa:

$$2S\lambda^2\sqrt{\beta^2-\lambda^2} = (\beta-\lambda^2)(\beta-\sqrt{\beta^2-\lambda^2}) \qquad /2.3/$$

gde je: $$\beta = \frac{2\Delta - E}{4SI} \qquad /2.\cdot/$$

$$\lambda = \cos\frac{Q}{2}$$

E: energija vezanih stanja

$$\Delta = \mu\mathcal{H} + 2S\ell I \qquad /2.5/$$

ℓ : broj dimenzija kristala

Q: zbir impulsa bozona $Q = |\vec{Q}| = |\vec{k}_1 + \vec{k}_2|$

Za spin $S = \frac{1}{2}$ dobijene su energije vezanih stanja dva bozona[21] /formule /2.2o/ i /2.26/:

$$E_1 = 0 \qquad /2.6/$$

$$E_2 = 2\Delta - 2I\left(1 - \frac{1}{2}\sin^2\frac{Q}{2}\right) \qquad /2.7/$$

identične sa vrednostima koje su ranije dobijene za vezana stanja dva magnona /na način koji je kritikovan u prvom odeljku ove glave/.

2. Marumori-jeva bozonska reprezentacija spinskih operatora može se okarakterisati kao neadekvatna jer daje dodatni energijski nivo $E=\frac{1}{2}$ koji ne može da se dobije pomoću drugih dvaju bozonskih reprezentacija. Naime, Marumori Yamamura i Tokunaga su predpostavili da je $S^z_f = 0$ za sve bozonske populacione brojeve veće od 2S pa je zbog ovog neprirodnog uslova dobijena bozonska reprezentacija neadekvatna za opis feromagneta.

3. Izračunata je talasna funkcija vezanog stanja dva bozona[23] u jednodimenzionalnom kristalu i u aproksimaciji najbližih suseda pomoću Hamiltonijana u kojem su spinski operatori predstavljeni preko bozonskih pomoću egzaktne bozonske reprezentacije[14]:

$$|2\rangle_B = \frac{1}{N\sqrt{2}} \sum_{fQ} \frac{e^{ifQ}}{\sqrt{1+2\left(\frac{\beta}{\lambda}\right)^2 \mathfrak{S}_Q^2}} \left\{ B_f^+ B_f^+ + \frac{\beta}{\lambda}\left[e^{i\frac{Q}{2}} B_f^+ B_{f+1}^+ + e^{-i\frac{Q}{2}} B_f^+ B_{f-1}^+ \right] \right\} |0\rangle \qquad /2.8/$$

gde je:

$$\mathfrak{S}_Q = \left[\frac{\beta}{\beta+\lambda^2(2S-1)} - \Omega \right]^{-1} \qquad /2.9/$$

$$\Omega = 1 - \sqrt{1 - \frac{1}{2S}} \qquad /2.10/$$

4. Takođe je dobijena talasna funkcija vezanih stanja dva magnona u Heisenberg-ovom jednodimenzionalnom feromagnetu sa spinom $S=\frac{1}{2}$ i u aproksimaciji najbližih suseda[23]:

$$|2\rangle_S = \frac{1}{N\sqrt{2}} \sum_{f\sqrt{2}} \frac{e^{iQf}}{\sqrt{1+2\left(\frac{\beta}{\lambda}\right)^2}} \left\{ S_f^- S_f^- + \frac{\beta}{\lambda}\left[e^{i\frac{Q}{2}} S_f^- S_{f+1}^- + e^{-i\frac{Q}{2}} S_f^- S_{f-1}^- \right] \right\} |0\rangle \qquad /2.11/$$

Ove funkcije sadrže član $S_+ S_+$ koji opisuje dva magnona na jednom čvoru. U slučaju spina $S = \frac{1}{2}$ $S_f^- S_f^- |0\rangle$ je identički ravno nuli pa bi to bio razlog da se ovaj član odbaci. Među-tim, ako se celo razmatranje počne sa $\left(S_f^-\right)^2 = 0$ tada nije moguće formirati jednačinu koja definiše vezana stanja u feromagnetu sa spinom $S = \frac{1}{2}$, jer koeficijenti A_{ii} mogu da budu proizvoljni te smo zato prisiljeni, da bi uopšte mogli da razmatramo problem, da predpostavimo $S_f^- S_f^- |0\rangle \neq 0$.

5. Pokazano je da Dyson-ova reprezentacija daje dvobozonsku talasnu funkciju vezanog stanja formalno identičnu sa talasnom funkcijom vezanog stanja dva magnona /2.11/.:

$$|2\rangle_D = \frac{1}{N\sqrt{2}}\sum_{Q,f}\frac{e^{iQf}}{\sqrt{1+2\left(\frac{\beta}{\lambda}\right)^2}}\Big\{B_f^+ B_f^+ + \frac{\beta}{\lambda}\Big[e^{i\frac{Q}{2}} B_f^+ B_{f+1}^+ + e^{-i\frac{Q}{2}} B_f^+ B_{f-1}^+\Big]\Big\}|0\rangle \qquad /2.12/$$

Ako se pak vezana stanja dva magnona u Heisenberg-ovom feromagnetu sa spinom $S = \frac{1}{2}$ tretiraju pomoću egzaktne bozonske reprezentacije dobija se talasna funkcija:

$$|2\rangle_{SB} = \frac{1}{N\sqrt{2}}\sum_{Qf}\frac{e^{iQf}}{\sqrt{1+2\left(\frac{\beta}{\lambda}\right)^2 R_\Omega^2}}\Big\{B_f^+ B_f^+ + \frac{\beta}{\lambda}R_\Omega\Big[e^{i\frac{Q}{2}} B_f^+ B_{f+1}^+ + e^{-i\frac{Q}{2}} B_f^+ B_{f-1}^+\Big]\Big\}|0\rangle \qquad /2.13/$$

gde je

$$R_\Omega = \frac{1}{1-\Omega} \qquad /2.14/$$

koja se razlikuje od talasnih funkcija /2.8/ i /2.12/.

6. Kao test ispravnosti različitih talasnih funkcija predložen je proces prelaska iz vezanih u slobodna stanja pod uticajem spo-

ljašnjeg periodičnog magnetnog polja.

Za razne vrednosti spina S feromagneta tabelirane su[23] verovatnoće prelaza iz dvobzonskih vezanih stanja u slobodna bozonska stanja i verovatnoće prelaza iz dvomagnonskih vezanih u slobodna magnonska stanja u jednodimenzionalnom slučaju. Na taj način trasiran je pravac istraživanja koje bi dalo bliže informacije o karakteru realnih elementarnih ekscitacija u feromagnetu. Potrebno je, naime, izabrati i tretirati teorijski pomoću bozonskog i spinskog formalizma takve dvočestične procese u feromagnetu koje je moguće eksperimentalno utvrditi i meriti. Na taj način bi eksperiment definitivno odlučio koji formalizam najadekvatnije opisuje realne situacije i procese u feromagnetu.

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Problem of Bound States in Ferromagnets with Spin $S=\frac{1}{2}$.

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Summary. — The problem of the bound states in the Heisenberg ferromagnet of spin $S=\frac{1}{2}$ is considered using the exact boson representation. The results are identical with those obtained with the spin operators but the present approach does not suffer from the inconsistencies inherent to the spin-operator approach. Also it is shown that other existing boson representations (Marumori, Dyson) are inadequate for the treatment of the bound states of the Heisenberg ferromagnet with spin $S=\frac{1}{2}$.

1. – Introduction.

In this paper the analysis of the Heisenberg ferromagnet in terms of the different representations for the spin operators will be extended to the problem of bound states. As was mentioned in (1), where a ferromagnet in a strong external magnetic field was considered, the influence of bound states on the thermodynamical properties at low temperatures would be expected to be important. Indeed, if $\mu\mathscr{H}\gg I$, where μ is the magnetic moment of the atom, $\mathscr{H}$ the external magnetic field and I the exchange integral (this is realizable in weak ferromagnets only), then the most important term in the ferromagnet with spin $S=\frac{1}{2}$ is

$$H_1(S)=\mu\mathscr{H}\sum_n\left(\tfrac{1}{2}-S_n^z\right), \tag{1.1}$$

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where S_n^z is the component of the spin in the direction of the z-axis. With the use of the Bose representation of the spin operators given in (2)

$$(1.2)\qquad S_n^+ = \left[\sum_{\nu=0}^{\infty}\frac{(-2)^\nu}{(1+\nu)!}B_n^{\dagger\nu}B_n^\nu\right]^{\frac{1}{2}}B_n\,;\qquad \frac{1}{2}-S_n^z=\sum_{\nu=0}\frac{(-2)^\nu}{(1+\nu)!}B_n^{\dagger\nu+1}B_n^{\nu+1}$$

($S_n^{\pm}=S_n^x \pm iS_n^y$ and $B_n^\dagger$ and B_n are Bose operators) one can easily obtain

$$(1.3)\qquad \big(H_1(B)\big)_2=\left(\mu\mathscr{H}\sum_n\sum_{\nu=0}^{\infty}\frac{(-2)^\nu}{(1+\nu)!}B_n^{\dagger\nu+1}B_n^{\nu+1}\right)_2=0\,.$$

The symbol $(...)_2$ denotes the expectation value of the operator in the bracket taken in a state with two bosons bound at one lattice point. This result suggested that the inclusion of the interaction terms proportional to the exchange integral could influence the possible existence of an observable low-lying level of energy $E_B \sim I^2/\mu\mathscr{H}$ corresponding to the two-boson bound state. Since the energy of free bosons is $E_e=\mu\mathscr{H}+\frac{1}{2}IK^2$ (K is the wave vector) the main contributions to the thermodynamical properties of the ferromagnets would come from the bound states due to their great population.

However it turned out that the inclusion of the interaction did not make « observable » the level $E_B=0$ (1.3), and the results of papers (3,4) (see also (5) Chap. V) were thus confirmed. We think that such a confirmation was necessary because the analysis of bound states which has been done in (3-5) in terms of the spin operators contained some elements that could cause doubts concerning the validity of the results and these are the use of the Fourier transform of spin operators, which is not a canonical transform and also the use of « non-existing » states $(S_n^-)^2|0\rangle$ in the case of spin $S=\frac{1}{2}$. Except this, the equality of the results in both pictures, namely in the spin and boson ones, confirms once more the boson character of the elementary excitations in ferromagnet.

Since the problem of bound states is exactly soluble, and since there are other Bose representations of spin operators (Marumori, Dyson), we used this problem to test the adequacy of these representations in analysis of the bound states. It was found that in this respect none of these other representations is adequate. In the case of Marumori's representation this inadequacy shows up in the appearance of a redundant bound state with an energy $E=\frac{1}{2}\Delta$. As to Dyson's Hamiltonian of ideal spin waves, it gives correct values

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for the bound-state energies, but the corresponding wave functions are not correct. This incorrectness yields wrong values for the transition probability between free magnons and the bound states. This is explicitly illustrated in the case of the transition between free and bound states influenced by an external radiofrequent magnetic field. Thus it turns out that the Dyson representation is adequate for the analysis of the thermodynamical properties of ferromagnets, but is inadequate for the analysis of its kinetic properties.

2. – Wave functions and energies of bound states.

As we stated in the Introduction, in the analysis of the bound states we shall use the exact Bose representation of spin operators. The spin-operator representation of the Hamiltonian of a simple cubic crystal has the following form in the nearest-neighbour approximation:

$$H(S) = E_0 + \Delta \sum_n (\tfrac{1}{2} - S_n^z) - \tfrac{1}{2} I \sum_{n\lambda} S_n^- S_{n+\lambda}^+ - \tfrac{1}{2} I \sum (\tfrac{1}{2} - S_n^z)(\tfrac{1}{2} - S_{n+\lambda}^z)\,, \tag{2.1}$$

where

$$E_0 = -\tfrac{1}{2} N\mu\mathscr{H} - \tfrac{1}{4} NlI\,, \qquad \Delta = \mu\mathscr{H} + lI\,,$$

N: the number of atoms in the crystal,

l: the number of dimensions of the crystal, and

$\boldsymbol{\lambda}$: the vector connecting the nearest neighbours.

By use of (1.2) we obtain the Hamiltonian of the system in the boson picture:

$$\begin{aligned} H(B) = E_0 + \Delta \sum_n \sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(1+\nu)!} B_n^{\dagger\nu+1} B_n^{\nu+1} - \\ -\frac{1}{2} I \sum_{n\lambda} B_n^\dagger \left[\sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(1+\nu)!} B_n^{\dagger\nu} B_n^\nu\right]^{\frac{1}{2}} \left[\sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(1+\nu)!} B_{n+\lambda}^{\dagger\nu} B_{n+\lambda}^\nu\right]^{\frac{1}{2}} B_{n+\lambda} - \\ -\frac{1}{2} I \sum_{n\lambda} \left[\sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(1+\nu)!} B_n^{\dagger\nu+1} B_n^{\nu+1}\right] \left[\sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(1+\nu)!} B_{n+\lambda}^{\dagger\nu+1} B_{n+\lambda}^{\nu+1}\right]. \end{aligned} \tag{2.2}$$

The wave function of the ground state of the boson system described by the Hamiltonian (2.2) is defined by

$$\psi_0 \equiv |0\rangle \tag{2.3}$$

and the wave function describing the state with two Bose excitations in the crystal by

$$\psi_2 = |2\rangle = \sum_{g_1 g_2} A_{g_1,g_2} B^\dagger_{g_1} B^\dagger_{g_2}|0\rangle . \tag{2.4}$$

The coefficients A_{g_1,g_2} are symmetrical, with $A_{g_1,g_1} \neq 0$.

Since we have

$$H(B)|2\rangle = \sum_{g_1 g_2} A_{g_1,g_2} H(B) B^\dagger_{g_1} B^\dagger_{g_2}|0\rangle = E_2 \sum_{g_1 g_2} A_{g_1,g_2} B^\dagger_{g_1} B^\dagger_{g_2}|0\rangle$$

and

$$\sum_{g_1 g_2} A_{g_1,g_2} B^\dagger_{g_1} B^\dagger_{g_2} H(B)|0\rangle = E_0 \sum_{g_1 g_2} A_{g_1,g_2} B^\dagger_{g_1} B^\dagger_{g_2}|0\rangle ,$$

where the second equation is obtained by left-applying the operator $\sum_{g_1 g_2} A_{g_1,g_2} B^\dagger_{g_1} B^\dagger_{g_2}$ to $H(B)|0\rangle = E_0|0\rangle$, subtraction of these two equations gives the equation

$$\sum_{g_1 g_2} A_{g_1,g_2}[B^\dagger_{g_1} B^\dagger_{g_2}, H(B)]|0\rangle = -E \sum_{g_1 g_2} A_{g_1,g_2} B^\dagger_{g_1} B^\dagger_{g_2}|0\rangle \tag{2.5}$$

determining the excitation energy of the crystal $E = E_2 - E_0 > 0$ and the coefficients A_{g_1,g_2} defining the excited state $|2\rangle$. By $[A, B]$ is denoted the commutator of the operators A and B. Substitution of (2.2) into (2.5) gives

$$\begin{aligned}\sum_{g_1,g_2} \{A_{g_1,g_2}[E - 2\Lambda(1-\delta_{g_1,g_2})] B^\dagger_{g_1} B^\dagger_{g_2} + \tfrac{1}{4} I \sum_{\lambda} [(A_{g_1+\lambda,g_2} + \\ + A_{g_1-\lambda,g_2} + A_{g_1,g_2+\lambda} + A_{g_1,g_2-\lambda} - 2\delta_{g_1,g_2} A_{g_1,g_1+\lambda} + \\ + 2\delta_{g_1,g_2} A_{g_1,g_1-\lambda}) B^\dagger_{g_1} B^\dagger_{g_2} - 2N^{-1}(A_{g_1+\lambda,g_1+\lambda} B^\dagger_{g_1} B^\dagger_{g_1+\lambda} + \\ + A_{g_1-\lambda,g_1-\lambda} B^\dagger_{g_1} B^\dagger_{g_1-\lambda} - A_{g_1,g_1+\lambda} B^\dagger_{g_1} B^\dagger_{g_1+\lambda} - A_{g_1,g_1-\lambda} B^\dagger_{g_1} B^\dagger_{g_1-\lambda})]\}|0\rangle = 0 .\end{aligned} \tag{2.6}$$

After Fourier transformations in (2.6)

$$\left\{\begin{aligned} A_{g_1,g_2} &= N^{-2} \sum_{K_1 K_2} \exp[i(K_1 g_1 + K_2 g_2)] A(K_1, K_2) , \\ B_g &= N^{-\frac{1}{2}} \sum_{K} \exp[iKg] B_K , \end{aligned}\right. \tag{2.7}$$

where K_1, K_2 and K are wave vectors, we obtain

$$\begin{aligned}[E - E_Q(q)]\alpha_Q(q) - 2IN^{-1} \sum_{q'\lambda} \cos q\lambda \left(\cos \frac{Q\lambda}{2} - \cos q'\lambda\right) \alpha_Q(q') = \\ = -N^{-1} \sum_{q'} E_Q(q')\, \alpha_Q(q') ,\end{aligned} \tag{2.8}$$

where

$$Q = K_1 + K_2\,, \tag{2.9a}$$

$$q_1 = \tfrac{1}{2}(K_1 - K_2)\,, \tag{2.9b}$$

$$A_{g_1,g_2} = N^{-2}\sum_{Qq}\exp\left[i\tfrac{1}{2}Q(g_1+g_2) + iq(g_1-g_2)\right]\alpha_Q(q)\,, \tag{2.9c}$$

$$E_Q(q) = 2\Delta - 2I\sum_{\lambda}\cos\frac{Q\lambda}{2}\cos q\lambda\,. \tag{2.9d}$$

In the expression (2.9d), which represents the sum of energies of free bosons and in eq. (2.8) also, $\boldsymbol{\lambda}$ goes over a half of the neighbours.

The energies of bound states are by the definition all values $E < E_Q(q)$ corresponding to the nontrivial solution $\alpha_Q(q) \neq 0$ of the integral equation (2.8). If we take in (2.8)

$$-N^{-1}\sum_{q} E_Q(q')\alpha_Q(q') = \beta_Q \tag{2.10}$$

we obtain

$$\alpha_Q(q) = \frac{\beta_Q}{E - E_Q(q)} + \frac{2IN^{-1}}{E - E_Q(q)}\sum_{q'\lambda}\cos q\lambda\left(\cos\frac{Q\lambda}{2} - \cos q\cdot\lambda\right)\alpha_Q(q')\,. \tag{2.11}$$

Iteration of (2.10) by means of (2.11) with the assumption $\beta_Q \neq 0$ gives

$$1 + Y_Q^{(l)} + \hat{A}_Q^{(l)}\hat{D}_Q^{(l)}\hat{B}_Q^{(l)} = 0\,. \tag{2.12}$$

The following notations are used in this equation:

$$\left.\begin{aligned} Y_Q^{(l)} &= \frac{1}{\pi^l}\int_0^\pi \mathrm{d}x_1 \dots \mathrm{d}x_l\,\frac{\eta - T_l}{T_l - \mu}\,,\\ T_l &= \sum_{\varrho=1}^{l}\cos\frac{Q_\varrho}{2}\cos x_\varrho\,,\\ \eta &= \frac{\Delta}{I}\,,\\ \mu &= \frac{2\Delta - E}{2I}\,, \end{aligned}\right. \tag{2.13a}$$

$$\left.\begin{aligned} \hat{A}_Q^{(l)} &= \left(A_Q^{(1)}A_Q^{(2)}\dots A_Q^{(l)}\right),\\ A_Q^{(\varrho)} &= \frac{1}{\pi^l}\int_0^\pi \mathrm{d}x_1 \dots \mathrm{d}x_l\,\frac{(\eta - T_l)\cos x_\varrho}{T_l - \mu}\,, \qquad \varrho = 1, 2, \dots, l\,, \end{aligned}\right. \tag{2.13b}$$

$$(2.13c)\quad \begin{cases} \hat{B}_Q^{(l)} = \begin{pmatrix} B_Q^{(1)} \\ \vdots \\ B_Q^{(l)} \end{pmatrix}, \\ B_Q^{(\varrho)} = \dfrac{1}{\pi^l}\displaystyle\int_0^{\pi} \mathrm{d}x_1 \dots \mathrm{d}x_l \, \frac{\cos(Q_\varrho/2) - \cos x_\varrho}{T_l - \mu}, \qquad \varrho = 1, 2, \dots, l, \end{cases}$$

$$(2.13d)\quad \begin{cases} \hat{D}_Q^{(l)} = (1 - \hat{C}_Q^{(l)})^{-1}, \qquad \hat{C}_Q^{(l)} = \begin{pmatrix} C_Q^{(11)} \dots C_Q^{(1l)} \\ \vdots \\ C_Q^{(l1)} \dots C_Q^{(ll)} \end{pmatrix}, \\ C_Q^{(\varrho\varrho')} = \dfrac{1}{\pi^l}\displaystyle\int_0^{\pi} \mathrm{d}x_1 \dots \mathrm{d}x_l \, \frac{(\cos Q_\varrho/2 - \cos x_\varrho)\cos x_{\varrho'}}{T_l - \mu}, \qquad \varrho, \varrho' = 1, \dots, l. \end{cases}$$

It has to be pointed out that the choice of β_Q made by (2.10) is the only possible choice ensuring the convergence of the expansion $\hat{A}_Q^{(l)}\hat{D}_Q^{(l)}\hat{B}_Q^{(l)}$ which is not evident if the matrices $\hat{A}_Q^{(l)}$, $\hat{B}_Q^{(l)}$ and $\hat{C}_Q^{(l)}$ are written in the form (2.13b)-(2.13d). The convergence of this expansion is explicitly verified by the diagonalization of the matrix $\hat{C}_Q^{(l)}$ by the unitary matrix $\hat{U}_Q^{(l)}$.

In the one-dimensional case the eq. (2.12) becomes

$$(2.14)\qquad (\eta - \mu)\left[\left(\cos\frac{Q}{2} - \mu\right) J_Q - 1\right] = 0,$$

where

$$(2.15)\qquad J_Q = \frac{1}{\pi}\int_0^{\pi} \frac{\mathrm{d}x}{\cos(Q/2)\cos x - \mu} = \frac{-1}{\sqrt{\mu^2 - \cos^2(Q/2)}}.$$

Assuming $\cos(Q_1/2) = \cos(Q_2/2) = L_2$ in the two-dimensional crystal, the formulae (2.13b)-(2.13d) reduce to

$$(2.16)\quad \begin{cases} A_Q^{(1)} = A_Q^{(2)} = \dfrac{1}{\pi^2}\displaystyle\int_0^{\pi}\int_0^{\pi} \mathrm{d}x_1\,\mathrm{d}x_2 \, \frac{\eta - L_2(\cos x_1 + \cos x_2)}{L_2(\cos x_1 + \cos x_2) - \mu} \cos x_1 = a^{(2)}, \\ B_Q^{(1)} = B_Q^{(2)} = \dfrac{1}{\pi^2}\displaystyle\int_0^{\pi}\int_0^{\pi} \mathrm{d}x_1\,\mathrm{d}x_2 \, \frac{L_2 - \cos x_1}{L_2(\cos x_1 + \cos x_2) - \mu} = b^{(2)}, \\ C_Q^{(11)} = C_Q^{(22)} = \dfrac{1}{\pi^2}\displaystyle\int_0^{\pi}\int_0^{\pi} \mathrm{d}x_1\,\mathrm{d}x_2 \, \frac{(L_2 - \cos x_1)\cos x_1}{L(\cos x_1 + \cos x_2) - \mu} = c^{(2)}, \\ C_Q^{(12)} = C_Q^{(21)} = \dfrac{1}{\pi^2}\displaystyle\int_0^{\pi}\int_0^{\pi} \mathrm{d}x_1\,\mathrm{d}x_2 \, \frac{(L_2 - \cos x_1)\cos x_2}{L_2(\cos x_1 + \cos x_2) - \mu} = d^{(2)}, \end{cases}$$

and (2.12) to

$$(2.17)\qquad (\eta-\mu)(1-c^{(2)}-d^{(2)})(1-c^{(2)}+d^{(2)})=0\,.$$

With the similar assumption $\cos(Q_1/2)=\cos(Q_2/2)=\cos(Q_3/2)=L_3$, we have for the three-dimensional lattice

$$(2.18)\quad \left\{\begin{aligned}
&A_Q^{(1)}=A_Q^{(2)}=A_Q^{(3)}=\frac{1}{\pi^3}\int_0^\pi\!\!\int_0^\pi\!\!\int_0^\pi \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\frac{\eta-L_3(\cos x_1+\cos x_2+\cos x_3)}{L_3(\cos x_1+\cos x_2+\cos x_3)-\mu}\cos x_1\equiv a^{(2)},\\
&B_Q^{(1)}=B_Q^{(2)}=B_Q^{(3)}=\frac{1}{\pi^3}\int_0^\pi\!\!\int_0^\pi\!\!\int_0^\pi \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\frac{L_3-\cos x_1}{L_3(\cos x_1+\cos x_2+\cos x_3)-\mu}\equiv b^{(3)},\\
&C_Q^{(11)}=C_Q^{(22)}=C_Q^{(33)}=\frac{1}{\pi^3}\int_0^\pi\!\!\int_0^\pi\!\!\int_0^\pi \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3\,\frac{(L_3-\cos x_1)\cos x_1}{L_3(\cos x_1+\cos x_2+\cos x_3)-\mu}\equiv c^{(3)},\\
&C_Q^{(12)}=C_Q^{(13)}=C_Q^{(21)}=C_Q^{(23)}=C_Q^{(31)}=C_Q^{(32)}\equiv d^{(3)},\\
&d^{(3)}=\frac{1}{\pi^3}\int_0^\pi\!\!\int_0^\pi\!\!\int_0^\pi \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}\,x_3\,\frac{(L_3-\cos x_1)\cos x_2}{L_3(\cos x_1+\cos x_2+\cos x_3)-\mu},
\end{aligned}\right.$$

while (2.12) reduces to

$$(2.19)\qquad (\eta-\mu)(1-c^{(3)}+d^{(3)})^2(1-c^{(3)}-2d^{(3)})=0\,.$$

On the basis of (2.14), (2.17) and (2.19) we can conclude the following:

a) From the condition

$$\eta-\mu=0\,,$$

which is the same for all $l=1,2,3$ it follows that

$$(2.20)\qquad E=0\,,$$

e.g. it is shown that an observable low-lying energy level does not exist; the situation in the crystal is the same as in the gas (see (1.3)).

b) The energies of the observable bound states which follow from

$$(2.21)\qquad \left(\cos\frac{Q}{2}-\mu\right)J_Q-1=0$$

in the one-dimensional case, from

$$1 - c^{(2)} + d^{(2)} = 0 \quad \text{and,} \quad 1 - c^{(2)} - d^{(2)} = 0 \tag{2.22}$$

in the two-dimensional case and from the conditions

$$(1 - c^{(3)} + d^{(3)})^2 = 0 \quad \text{and} \quad (1 - c^{(3)} - 2d^{(3)}) = 0 \tag{2.23}$$

in the three-dimensional case, are identical with the energies given in (3-5). Consequently, the results from (3-5) are in this way confirmed by means of the Bose representation (1.2).

We shall not analyse the results obtained here because the detailed analysis is presented in (4,5). Instead of this, we shall determine $\alpha_Q(q)$ from the eq. (2.8) using (2.20) and (2.21). Bound-state wave functions will be evaluated with the help of functions $\alpha_Q(q)$ and the expressions (2.4) and (2.9*c*). We shall investigate in details the one-dimensional case only since in other cases it is not possible to find $\alpha_Q(q)$ in an explicit analytical form.

Let us consider first the solution $E = 0$. Substituting (2.20) into (2.8) we find that in this case for all $l = 1, 2, 3$, the function $\alpha_Q(q)$ is

$$\alpha_Q(q) = \gamma_Q \eta^{-1}, \tag{2.24}$$

where γ_Q is an arbitrary function of Q. From (2.24) and (2.9*c*) it follows that

$$A_{g_1,g_2} = \delta_{g_1,g_2} N^{-1} \sum_Q \exp\left[\frac{i}{2} Q(g_1 + g_2)\right] \gamma_Q .$$

The properly normalized wave function is in this case expressed by

$$\left\{ \begin{aligned} & |2_Q\rangle_{E=0} = \sum_g v_g B_g^{+2} |0\rangle , \\ & v_g = \frac{1}{N\sqrt{2}} \sum_Q \exp[iQg] . \end{aligned} \right. \tag{2.25}$$

We shall see later that the state (2.25) takes no part in any physical process in the crystal including the spin-phonon interaction and the interaction of the ferromagnetic with the external periodical magnetic field, *e.g.* it is unobservable even in this sense.

From the eq. (2.21) we obtain for the bound-state energy

$$\widetilde{E} = 2\Delta - 2I\left(1 - \frac{1}{2}\sin^2\frac{Q}{2}\right). \tag{2.26}$$

The substitution of (2.26) into (2.8) gives the equation

$$\left(\nu_Q-\cos\frac{Q}{2}\cos q\right)\alpha_Q(q)+\cos q\cdot N^{-1}\sum_{q'}\left(\cos\frac{Q}{2}-\cos q'\right)\alpha_Q(q')= \\ =N^{-1}\sum_{q'}\left(\eta-\cos\frac{Q}{2}\cos q'\right)\alpha_Q(q'), \tag{2.27}$$

where

$$\nu_Q=1-\frac{1}{2}\sin^2\frac{Q}{2}. \tag{2.28}$$

We are looking for a solution of (2.27) in the form

$$\alpha_Q(q)=X_Q(q)+Z_Q, \tag{2.29}$$

where

$$X_Q(q)=\frac{\gamma_Q\cos q}{\nu_Q-\cos(Q/2)\cos q} \tag{2.30}$$

is the solution of the homogeneous equation

$$\left(\nu_Q-\cos\frac{Q}{2}\cos q\right)\alpha_Q(q)+N^{-1}\cos q\sum_{q'}\left(\cos\frac{Q}{2}-\cos q'\right)\alpha_Q(q')=0. \tag{2.31}$$

If we substitute (2.29) into (2.27), passing from sums to integrals, we have

$$\alpha_Q(q)=\gamma_Q\left(\frac{\cos q}{\nu_Q-\cos(Q/2)\cos q}-\frac{2\cos(Q/2)}{\sin^2(Q/2)}\right). \tag{2.32}$$

On account of (2.32) and (2.9c) it is easy to obtain

$$A_{g,g}=0\,;\qquad A_{g_1,g_2}=\frac{1}{N\pi}\sum_Q\gamma_Q\exp\left[\frac{i}{2}Q(g_1+g_2)\right]R_Q(g_1-g_2), \tag{2.33}$$

where

$$R_Q(g_1-g_2)=\frac{1}{\cos(Q/2)}\sum_{\nu=0}^{\infty}\left(\frac{\cos Q/2}{\nu_Q}\right)^{\nu}\int_0^{\pi}\cos(g_1-g_2)q\cos^{\nu}q\,\mathrm{d}q. \tag{2.34}$$

The value of the integral in (2.34) is given in ref. (6), integral no. 3.631.17.

(6) I. S. Gradstein and I. M. Ryžik: *Tables of Integrals, Sums and Products* (Moscow, 1963) (in Russian).

If we assume that the bound states are formed by coupling of the nearest neghbours, the equation (2.33) gives

$$A_{g,g\pm 1} = \frac{1}{N}\sum_Q \gamma_Q \exp\left[iQ\left(g \pm \frac{1}{2}\right)\right] G(Q)\,, \tag{2.35}$$

where

$$G(Q) = \frac{1+\cos^2(Q/2)}{\sin^2(Q/2)}\,.$$

The normalized wave function corresponding to the energy (2.26) is

$$|2_Q\rangle_{E-\tilde{E}} = \sum_g (\Gamma_{g,g+1} B_g^\dagger B_{g+1}^\dagger + \Gamma_{g,g-1} B_g^\dagger B_{g-1}^\dagger)|0\rangle\,, \tag{2.36}$$

where

$$\Gamma_{g,g\pm 1} = \frac{1}{2N}\sum_Q \exp\left[iQ\left(g \pm \frac{1}{2}\right)\right]\,. \tag{2.37}$$

The states $|2_Q\rangle_{E-\tilde{E}}$ and $|2_Q\rangle_{E-0}$ are orthogonal. We have to note that the normalization is accurate to the arbitrary additive function $F(Q)$ orthogonal to $G^2(Q)$, *e.g.*

$$\sum_Q F(Q)\, G^2(Q) = 0\,,$$

but $F(Q)$ gives no contribution to any measurable physical quantity on account of the orthogonality condition.

By an analogous treatment we can find in two and in three-dimensional crystals the function $\alpha_Q(q)$ and the coefficients A_{g_1,g_2}. It is easy to prove that the coefficients A_{g_1,g_1} are equal to zero for all nonzero bound-state energies.

3. – The transitions under the influence of the external periodical magnetic field. The comparison with the results of other theories.

We have seen that in ferromagnet there are besides the free states

$$|1_K\rangle_{E-E_0} = \sum_g u_g B_g^\dagger|0\rangle\,, \quad u_g = N^{-\frac{1}{2}} \exp[iKg]\,, \quad E_0 = \Delta - I\sum_\lambda \cos q\lambda\,, \tag{3.1}$$

the bound states too. Therefore it is of interest to investigate the interaction of the spins with the lattice vibrations and with external periodical magnetic field, since these interactions could influence the transitions from the free to the bound states, and also the transitions between the bound states.

The Hamiltonian of the spin-phonon interaction has the following form ((7)):

$$(3.2)\quad H_{sp}=\sum_{Kn_1n_2}\varphi_K(n_1-n_2)(S^z_{n_1}S^z_{n_2}+S^-_{n_1}S^+_{n_2})(b_K-b^\dagger_K)\cdot(\exp[iKn_1]-\exp[iKn_2])\,,$$

where $b^\dagger_K$ and b_K are phonon creation and annihilation operators and

$$(3.3)\qquad \phi_K(n_1-n_2)=i\sqrt{\frac{1}{2N\omega_K}}\,(e_K\cdot\nabla I(n_1-n_2))\,.$$

Here ω_K represents the phonon energy and e_K its polarization. Using the exact Bose representation (1.2) we obtain

$$(3.4)\quad S^z_{n_1}S^z_{n_2}+S^-_{x_1}S^+_{n_2}=\left[\frac{1}{2}-\sum_{\nu=0}^{\infty}\frac{(-2)^\nu}{(1+\nu)!}B^{\dagger\nu+1}_{n_1}B^{\nu+1}_{n_1}\right]\left[\frac{1}{2}-\sum_{\nu=0}^{\infty}\frac{(-2)^\nu}{(1+\nu)!}B^{\dagger\nu+1}_{n_2}B^{\nu+1}_{n_2}\right]+$$
$$+B^\dagger_{n_1}\left[\sum_{\nu=0}^{\infty}\frac{(-2)^\nu}{(1+\nu)!}B^{\dagger\nu}_{n_1}B^{\nu}_{n_1}\right]^{\frac{1}{2}}\left[\sum_{\nu=0}^{\infty}\frac{(-2)^\nu}{(1+\nu)!}B^{\dagger\nu}_{n_2}B^{\nu}_{n_2}\right]^{\frac{1}{2}}B_{n_2}\equiv\hat{\varphi}(B)\,.$$

On the basis of the formulae (2.25), (2.36), (3.1) and (3.4) it is easy to obtain

$$(3.5)\qquad \begin{cases}\langle 1_K|_{E-E_c}\hat{\varphi}(B)|2_Q\rangle_{E-\tilde{E}}=0\,,\\ \langle 1_K|_{E-E_c}\hat{\varphi}(B)|2_Q\rangle_{E-0}=0\,,\\ \langle 2_Q|_{E-0}\,\hat{\varphi}(B)|2_Q\rangle_{E-\tilde{E}}=0\,,\end{cases}$$

i.e. we conclude that the spin-phonon interaction can not influence the transitions from free to the bound states and the transitions between the bound states.

The Hamiltonian of the interaction between spins and the periodical magnetic field is (see (7)):

$$(3.6)\qquad H_{sh}=-\mu\sum_{g\Omega}\exp[-i\Omega t]\,[\tfrac{1}{2}h^{(-)}_g(\Omega)S^+_g+\tfrac{1}{2}h^{(+)}_gS^-_g+h^{(z)}_gS^z_g]\,,$$

where Ω are the energies (the field frequencies) and $h^{(\pm)}_g(\Omega)$ and $h^{(z)}_o(\Omega)$ the field components in an energy representation.

For the matrix elements of the Hamiltonian $H_{sh}(B)$ which is obtained from H_{sh} by the use of (1.2) we have

$$(3.7)\qquad \begin{cases}\langle 2_Q|_{E-0}H_{sh}(B)|1_K\rangle_{E-E_c}=0\,,\\ \langle 2_Q|_{E-0}H_{sh}(B)|2_Q\rangle_{E-\tilde{E}}=0\,.\end{cases}$$

(7) S. V. Tyablikov: *The Methods of Quantum Theory in Magnetism* (Moscow, 1965) (in Russian).

From (3.5) (see the last two matrix elements) and (3.7) it is clear that the state $|2_Q\rangle_{\tilde{E}=0}$ is completely unobservable because it does not appear in any physical process in crystal.

The only possible transition under the influence of the Hamiltonian $H_{sh}(B)$ is

$$\langle 1_K|_{E=E_e} \to |2_Q\rangle_{E=\tilde{E}} .$$

If we confine ourselves to the one-dimensional crystal, when

$$E_e = \Delta - I \cos k \tag{3.8}$$

we get for the transition matrix element

$$\begin{aligned} M_{\tilde{E}>E_e}(t) &= \langle 1_K|_{E=E_e} H_{sh}(B)|2_Q\rangle_{E=\tilde{E}} = \\ &= -\frac{\mu}{2}\sum_{\Omega}\exp[-i\varepsilon t]\sum_{g}\{[h_{g+1}^{(-)}(\Omega)u_g^* + h_g^{(-)}(\Omega)u_{g+1}^*]\Gamma_{g,g+1} + \\ &\qquad + [h_{g-1}^{(-)}(\Omega)u_g^* + h_g^{(-)}(\Omega)u_{g-1}^*]\Gamma_{g,g-1}\} , \end{aligned} \tag{3.9}$$

where

$$\varepsilon = \Omega + E_e - \tilde{E} . \tag{3.9a}$$

If $\mu\mathcal{H} < I$ we have the transition $\langle 2_Q|_{E=\tilde{E}} \to |1_K\rangle_{E=E_e}$ and E_e and $\tilde{E}$ change the sign in (3.9a). When $h(\Omega)$ does not depend on the lattice point (this assumption is correct for the values of the wave length much larger than the dimensions of the crystal) the matrix element (3.9) for the given frequency Ω is expressed by

$$M_\Omega(K, t) = -\frac{\mu \cos K/2}{\sqrt{N}} \exp[i[\Omega - (\tilde{E} - E_e)]t]\, h^{(-)}(\Omega) \tag{3.10}$$

and the transition probability (summed over all the values of K) by

$$W = \mu^2 \int_{-\pi}^{+\pi} \cos^2\frac{K}{2}\, |h^{(-)}(E_K)|^2\, dk , \tag{3.11}$$

where

$$E_K = \Delta - 2I\left(1 - \frac{1}{2}\cos^2\frac{K}{2}\right) . \tag{3.12}$$

In conclusion of this Section we shall consider the results obtained with some other Bose representations of spin operators.

The Marumori's representation (see ([8])) gives the following effective boson Hamiltonian for the investigation of bound states:

$$H_M^{\text{eff}} = E_0 + \Delta \sum_n B_n^\dagger B_n - \tfrac{1}{2} I \sum_{n\lambda} B_n^\dagger B_{n+\lambda} - \tfrac{3}{4}\Delta \sum_n B_n^\dagger B_n^\dagger B_n B_n + \tag{3.13}$$

$$+ \tfrac{1}{2} I \sum_{n\lambda} B_n^\dagger B_n^\dagger B_n B_{n+\lambda} + \tfrac{1}{2} I \sum_{n\lambda} B_n^\dagger B_{n+\lambda}^\dagger B_{n+\lambda} B_{n+\lambda} - \tfrac{1}{2} I \sum_{n\lambda} B_n^\dagger B_{n+\lambda}^\dagger B_n B_{n+\lambda}\,.$$

Completely analogous treatment as in Sect. **2** leads, with the Hamiltonian (3.13), to the integral equation

$$[E - E_Q(q)]\alpha_Q(q) - \frac{2I}{N}\sum_{q'\lambda} \cos q\lambda \left(\cos\frac{Q\lambda}{2} - \cos q'\lambda\right)\alpha_Q(q') = \tag{3.14}$$

$$= -\frac{1}{N}\sum_{q'}\left[E_Q(q') - \frac{1}{2}\Delta\right]\alpha_Q(q')\,.$$

It is easy to show, using the results of Sect. **2** that the eq. (3.14) gives for the bound-state energies the values obtained from (2.9), but the level $E=0$ becomes here observable, *i.e.*

$$E = \tfrac{1}{2}\Delta\,. \tag{3.15}$$

The appearance of the additional level $E=\frac{1}{2}\Delta$ is the consequence of the imposed condition on the Marumori's representation. Namely, it is assumed in ([8]) that S_n^z vanishes for all boson occupation numbers which are larger than $2S$. In this way an inadequate boson picture of the ferromagnetic was obtained.

In the theory of magnetism it is often used the Dyson's Hamiltonian of the ideal spin waves (see ([9])):

$$H_D = E_0 + \Delta \sum_n B_n^\dagger B_n - \tfrac{1}{2} I \sum_{n\lambda} B_n^\dagger B_{n+\lambda} + \tag{3.16}$$

$$+ \tfrac{1}{2} I \sum_{n\lambda} B_n^\dagger B_{n+\lambda}^\dagger B_{n+\lambda} B_{n+\lambda} - \tfrac{1}{2} I \sum_{n\lambda} B_n^\dagger B_{n+\lambda}^\dagger B_n B_{n+\lambda}\,.$$

It is known that the effects of interactions between free bosons at low temperatures and for small magnetic fields are well described by the Hamiltonian (3.16). The integral equation defining the bound boson states in the system

(8) T. Marumori, M. Yamamura and G. Tokunaga: *Progr. Theor. Phys.* (*Kyoto*), **31**, 1009 (1964). See also: S. C. Pang, G. Klein and R. M. Dreizler *Ann. of Phys.*, **49**, 477 (1968).

(9) F. J. Dyson: *Phys. Rev.*, **102**, 1217 (1956). See also: S. V. Maleev: *Žurn. Èksp. Teor. Fiz.*, **33**, 1010 (1957).

described by the Hamiltonian (3.16) is

$$(3.17)\qquad [E-E_Q(q)]\alpha_Q(q)-\frac{2I}{N}\sum_{q'\lambda}\cos q\lambda\left(\cos\frac{Q\lambda}{2}-\cos q'\lambda\right)\alpha_Q(q')=0$$

and it gives the same values for energies as (2.8), except for the level $E=0$. Consequently, the Hamiltonian (3.16) gives the correct values for the bound-state energies but the wave functions obtained here differ from those following from (3.17). We shall demonstrate this on the one-dimensional lattice. For the energy $E=\widetilde{E}$ (2.26) the eq. (3.17) becomes in this case

$$(3.18)\qquad \left(\nu_Q-\cos\frac{Q}{2}\cos q\right)\alpha_Q(q)+\cos q\,\frac{1}{N}\sum_{q'}\left(\cos\frac{Q}{2}-\cos q'\right)\alpha_Q(q')=0\,.$$

The solution of eq. (3.18) is

$$(3.19)\qquad \alpha_Q(q)=\frac{\gamma_Q\cos q}{\nu_Q-\cos(Q/2)\cos q}$$

and the coefficients are

$$(3.19a)\qquad \left\{\begin{aligned} A_{g,g}&=\frac{1}{N}\sum_Q\gamma_Q\exp[iQg]\frac{1}{N}\sum_q\frac{\cos q}{\nu_Q-\cos(Q/2)\cos q}\,,\\ A_{g_1,g_2 \atop g_1\neq g_2}&=\frac{1}{N}\sum_Q\gamma_Q\exp\left[\frac{i}{2}Q(g_1+g_2)\right]\frac{1}{N}\sum_q\frac{\cos q\exp[iq(g_1-g_2)]}{\nu_Q-\cos(Q/2)\cos q}\,.\end{aligned}\right.$$

In the nearest-neighbour approximation the normalized wave function is

$$(3.20)\qquad |2'_Q\rangle_{E=\widetilde{E}}=\sum_g w_g B_g^{+2}|0\rangle+\sum_g(A_{g,g+1}B_g^\dagger B_{g+1}^\dagger+A_{g,g-1}B_g^\dagger B_{g-1}^\dagger)|0\rangle\,,$$

where

$$(3.21)\qquad w_g=\frac{1}{2N}\sum_Q\frac{G_1(Q)\exp[iQg]}{\sqrt{\frac{1}{2}G_1^2(Q)+G_2^2(Q)}}\,,$$

$$(3.22)\qquad A_{g,g\pm1}=\frac{1}{2N}\sum_Q\frac{G_2(Q)\exp\left[iQ(g\pm\frac{1}{2})\right]}{\sqrt{\frac{1}{2}G_1^2(Q)+G_2^2(Q)}}$$

and

$$(3.23)\qquad \left\{\begin{aligned} G_1(Q)&=\frac{2\cos(Q/2)}{\sin^2(Q/2)}\,,\\ G_2(Q)&=\frac{1+\cos^2(Q/2)}{\sin^2(Q/2)}\,.\end{aligned}\right.$$

As we see, the wave function includes the part which corresponds to the unphysical states with two bosons at the same lattice node. This changes the values of the physical quantities evaluated with such wave function. For instance the probability transition from $|2_Q\rangle_{E-\tilde{E}}$ to $|1_K\rangle_{E-E_0}$ under the influence of the external periodical magnetic field is

$$W_D = \mu^2 \int_{-\pi}^{+\pi} \cos^2\frac{K}{2}\, \frac{G_1^2(K)}{\frac{1}{2}G_1^2(K)+G_2^2(K)}\, |h^{(-)}(E_K)|^2 \, dk = \tag{3.24}$$

$$= \mu^2 \int_{-\infty}^{+\pi} \cos^2\frac{K}{2}\, \frac{(1+\cos^2(K/2))^2}{(1+\cos^2(K/2))^2 + 2\cos^2(K/2)}\, |h^{(-)}(E_K)|^2 \, dk$$

and it follows that the admixture of the unphysical states in $|2'_Q\rangle_{E-\tilde{E}}$ decreases the transition probability. If we suppose that the function $|h^{(-)}(\Omega)|^2$ is independent of Ω the ratio of the probabilities (3.11) and (3.24) is

$$\frac{W}{W_D} = 1.46\,. \tag{3.25}$$

4. – Conclusion.

In conclusion we emphasize the following:

a) the energies and the wave functions of the observable bound states are identical in the boson picture and in the paulion (spin) picture,

b) there is in the boson picture the completely unobservable state with two bosons bounded at the same lattice point, corresponding to the nonexisting state with two paulions at one lattice point,

c) Marumori's boson representation of spin operators was found inadequate since it contains an additional observable level which is not obtainable in the other, spin and boson (1.2), representations,

d) Dyson's Hamiltonian of the ideal spin waves gives the correct values of the bound states energies but wrong wave functions (namely, the wave functions include the unphysical states describing two bosons at the lattice point) leading to inaccurate results for the transition probabilities.

* * *

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RIASSUNTO (*)

Usando l'esatta rappresentazione dei bosoni, si studia il problema degli stati legati nel ferromagnete di Heisenberg di spin $S=\frac{1}{2}$. I risultati sono identici a quelli ottenuti con gli operatori di spin ma il metodo d'approccio qui usato non risente delle incoerenze inerenti al metodo degli operatori di spin. Si dimostra anche che altre rappresentazioni esistenti di bosoni (Marumori, Dyson) sono inadeguate alla trattazione degli stati legati del ferromagnete di Heisenberg di spin $S=\frac{1}{2}$.

(*) *Traduzione a cura della Redazione.*

Проблема связанных состояний в ферромагнетиках со спином $S=\frac{1}{2}$.

Резюме (*). — Используя точное бозонное представление, рассматривается проблема связанных состояний в гайзенберговском ферромагнетике со спином $S=\frac{1}{2}$. Результаты являются идентичными результатам, полученным с помощью операторов спина, но в настоящем подходе не существует противоречий, свойственных подхоху спиновых операторов. Также показывается, что другие существующие бозонные представления (Марумори, Дайсона) неадекватны для рассмотрения связанных состояний гайзенберговского ферромагнетика со спином $S=\frac{1}{2}$.

(*) *Переведено редакцией.*

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HOW TO ANALYSE BOUND STATES IN FERROMAGNETS WITH SPIN $S=\frac{1}{2}$

by

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1. INTRODUCTION

The problem of bound states in the Heisenberg ferromagnet (the bound state in ferromagnet is the collective excitation which is the product of two synchronous oscilations of the spin pairs) was considered by many autors |1, 2, 3|. In these papers the spin Hamiltonian or its Bose equivalent according to Dyson |3| was used for the spin of the arbitrary magnitude. For the case S = 1/2 the results are found by the supstitution of the value S = 1/2 in general formulae. If we should treat the ferromagnet with spin S = 1/2 in the same way as in the papers |1, 2, 3|, then the equation for the bound states could not be established! We shall try to demonstrate this statement in the following way.

The wave function describing two excitations in ferromagnet with spin S = 1/2 has in general the following form

$$|2>_S = \sum_{\vec{n},\vec{m}} C_{\vec{n},\vec{m}} S^-_{\vec{n}} S^-_{\vec{m}} |0> \qquad (1)$$

where $\vec{n}$ and $\vec{m}$ are the crystal lattice vectors, $S^-_{\vec{n}}$ - the operator which diminish the value of the z-projection of the spin for one unit and $C_{\vec{n}\vec{m}}$ the coefficients determing the amplitude of the probability that the spin at the points $\vec{n}$ and $\vec{m}$ will be changed for the unit. In the case S = 1/2 the state $(S^-_n)^2|0>=0$, i.e. in the summ (1) the term which corresponds to $\vec{n}=\vec{m}$ has to be "dropped" out. It is easy to show that this, completely regular treatment, leads to the bound state's energy of the value whych is about twice and a half

+ On leave from the Department of Physics at the University of Novi Sad, Novi Sad greater than the summ of the energies of two free magnons; naturally this is physically non-sence. On the other hand, if we keep these states in the summ (1), it is clear that we can multiply them with arbitrary coefficient, and this must not change the physical result because these states are equal to zero. However, for different values of the coefficients we get different results. We have seen in the given example that the result has no physical sence if these coeficients are equal to zero. Therefore the bound states in ferromagnets with spin $S = 1/2$ have to be investigated in some other representation. We shall use in this paper the Bose representation of the spin operators |4| but we shall give also the results of the Dyson's and Marumori's representation.

2. WAVE FUNCTIONS AND ENERGIES OF BOUND STATES

In the analysis of the bound states of two spins in Bose representation it is enough to use the formulae from |4| in the following approximation:

$$\frac{1}{2} - S^z_{\vec{n}} = B^+_{\vec{n}} B_{\vec{n}} - (B^+_{\vec{n}})^2 B^2_{\vec{n}}; \quad S^+_{\vec{n}} = B_{\vec{n}} - B^+_{\vec{n}} B^2_{\vec{n}}; \quad S^-_{\vec{n}} = B^+_{\vec{n}} - (B^+_{\vec{n}})^2 B_{\vec{n}} \tag{2}$$

The Hamiltonian of Heisenberg ferromagnet with spin $S = 1/2$ in Bose representation (2) and with nearest neighbour approximation has the form

$$H = E_0 + \Delta \sum_{\vec{n}} B^+_{\vec{n}} B_{\vec{n}} - \frac{I}{2} \sum_{\vec{n},\vec{\lambda}} B^+_{\vec{n}} B_{\vec{n}+\vec{\lambda}} + \frac{I}{2} \sum_{\vec{n},\vec{\lambda}} B^+_{\vec{n}} B^+_{\vec{n}+\vec{\lambda}} B_{\vec{n}+\vec{\lambda}} B_{\vec{n}+\vec{\lambda}} -$$

$$- \frac{I}{2} \sum_{\vec{n},\vec{\lambda}} B^+_{\vec{n}} B^+_{\vec{n}+\vec{\lambda}} B_{\vec{n}+\vec{\lambda}} B_{\vec{n}} - \Delta \sum_{\vec{n}} B^+_{\vec{n}} B^+_{\vec{n}} B_{\vec{n}} B_{\vec{n}} + \frac{I}{2} \sum_{\vec{n}} B^+_{\vec{n}} B^+_{\vec{n}} B_{\vec{n}} B_{\vec{n}+\vec{\lambda}} \tag{3}$$

where $E_0 = \frac{1}{2} \mu N \mathcal{H} - \frac{1}{4} N l I; \quad \Delta = \mu \mathcal{H} + l I$

- N - the number of atoms in the crystal
- l - the number of dimensions of the crystal
- μ - the magnetic moment of the atom
- $\mathcal{H}$ - the external magnetic field
- $\vec{\lambda}$ - the vector connecting the nearest neighbours
- I - the exchange integral for the nearest neighbours.

The wave function describing the system with two boson excitations is

$$\psi_2 = |2\rangle = \sum_{\vec{n},\vec{m}} A_{\vec{n},\vec{m}} B^+_{\vec{n}} B^+_{\vec{m}} |0\rangle \tag{4}$$

The equation determining the coefficients

$$\sum_{\vec{n},\vec{m}} A_{\vec{n},\vec{m}} \left\{ E + [B^+_{\vec{n}} B_{\vec{m}}, H] \right\} |0\rangle = 0$$

(where $E = E_2 - E_0$ is the difference of the bound state energy and the energy of the excited state) can be reduced to

$$(E - 2\Delta) A_{\vec{n},\vec{m}} + \frac{I}{2} \sum_{\vec{\lambda}} (A_{\vec{n},\vec{m}+\vec{\lambda}} + A_{\vec{n}+\vec{\lambda},\vec{m}}) = -2\Delta A_{\vec{m},\vec{n}} \delta_{\vec{n},\vec{m}} +$$

$$+ I \sum_{\vec{\lambda}} A_{\vec{n},\vec{m}+\vec{\lambda}} \delta_{\vec{n},\vec{m}} + I \sum_{\vec{\lambda}} A_{\vec{n},\vec{n}} \delta_{\vec{n},\vec{m}+\vec{\lambda}} - I \sum_{\vec{\lambda}} A_{\vec{n},\vec{n}} \delta_{\vec{n},\vec{m}+\vec{\lambda}} \tag{5}$$

After the Fourier transformation in (5)

$$A_{\vec{n},\vec{m}} = \frac{1}{N^2} \sum_{\vec{Q},\vec{q}} a_{\vec{Q}}(\vec{q}) e^{i\vec{Q}\vec{R} + i\vec{q}\vec{r}} \; ; \; \vec{R} = \frac{\vec{n}+\vec{m}}{2}, \; \vec{r} = \vec{n} - \vec{m} \tag{6}$$

where $\vec{Q}$, $\vec{q}$ are the wave vectors, we obtain the homogenous Fredholm's equation with degenerate kernel:

$$a_{\vec{Q}}(\vec{k}) = \frac{1}{(2\pi)^\ell} \underbrace{\int_{-\pi}^{+\pi} \cdots \int_{-\pi}^{+\pi}}_{\ell \text{ times}} \left\{ \frac{-\eta + L \sum_{i=1}^{\ell} \cos k_i}{-\mu + L \sum_{i=1}^{\ell} \cos k_i} + \frac{\sum_{i=1}^{\ell} (L - \cos k_i) \cos q_i}{-\mu + L \sum_{i=1}^{\ell} \cos k_i} \right\} a_{\vec{Q}}(\vec{q}) \, d^3 q_1 \ldots d^3 q_\ell \tag{7}$$

where $\eta = \frac{\Delta}{I}$, $\mu = \frac{2\Delta - E}{2I}$, $L = \cos\frac{Q_1}{2} = \cos\frac{Q_2}{2} = \cdots = \cos\frac{Q_\ell}{2}$

In the further treatment we shall confine ourselves to the onedimensional case when the solutions can be given explicitly. The equation (7) has the nontrivial solutions for

$$E_1 = 0 \quad \text{and} \quad E_2 = 2\Delta - 2I\left(1 - \frac{1}{2}\sin^2\frac{Q}{2}\right) \tag{8}$$

and the corresponding wave functions are respectively

$$|2_Q\rangle_{E=E_1} = \sum_{n} \frac{1}{N\sqrt{2}} \sum_{Q} e^{iQn} (B^+_n)^2 |0\rangle \tag{9}$$

$$|2_Q\rangle_{E=E_2} = \sum_n \frac{1}{2N} \sum_Q \left\{ e^{iQ(n+\frac{1}{2})} B_n^+ B_{n+1}^+ + e^{iQ(n-\frac{1}{2})} B_n^+ B_{n-1}^+ \right\} |0\rangle \qquad (10)$$

where (10) is obtained under the assumption that only the nearest neighbours are coupled.

In this way we have shown that the state (9) with two bosons bound at one lattice point is undiserrable ($E=0$) or unphysical, as it is often called. The energy E_2 has been obtained also by the other autors |1, 2, 3| but in the way described in Introduction. Here we get this result in the complitely regular manner.

3. THE COMPARASION WITH THE RESULTS OF OTHER BOSE REPRESENTATIONS

Analogous calculation with Dyson's representation of the spin operators

$$S_n^z = B_n^+ B_n - \frac{1}{2} ; \; S_n^- = B_n - B_n^+ B_n B_n ; \; S_n^+ = B_n^+ \qquad (11)$$

gives the same result for the bound state energy

$$E' = 2\Delta - 2I\left(1 - \frac{1}{2}\sin^2\frac{Q}{2}\right)$$

but the wave function contains the "mixtures" of the unphysical states describing two bosons bound at one lattice point, i.e. it is incorrect:

$$|2_Q\rangle_{E=E'} = \sum_n W_n (B_n^+)^2 |0\rangle + \sum_n \left(\Lambda_{n,n+1} B_n^+ B_{n+1}^+ + \Lambda_{n,n-1} B_n^+ B_{n-1}\right)|0\rangle$$

where

$$W_n = (2N)^{-1} \sum_n \frac{G_1 \exp(iQn)}{\left(\frac{1}{2}G_1^2 + G_2^2\right)^{1/2}} ; \; \Lambda_{n\pm1} = \frac{1}{2N} \sum_n \frac{G_2 \exp\left(iQ(n\pm\frac{1}{2})\right)}{\left(\frac{1}{2}G_1^2 + G_2^2\right)^{1/2}} \qquad (12)$$

$$G_1(Q) = \frac{2\cos\frac{Q}{2}}{\sin^2\frac{Q}{2}} , \quad G_2(Q) = \frac{1+\cos^2\frac{Q}{2}}{\sin^2\frac{Q}{2}}$$

It is easily to show that this incorrectness in the wave function decreases the transition probability from the bound to free states under the influence of the external periodical magnetic field for about 33% (see details in |5|).

From the Marumori's representation of the spin operators

$$\frac{1}{2} - S^z_{\vec{n}} = B^+_{\vec{n}} B_{\vec{n}} - \frac{3}{4}(B^+_{\vec{n}})^2 B_{\vec{n}}$$

$$S^+_{\vec{n}} = B_{\vec{n}} - B^+_{\vec{n}}(B_{\vec{n}})^2 \; ; \; S^-_{\vec{n}} = B^+_{\vec{n}} - (B^+_{\vec{n}})^2 B_{\vec{n}} \qquad (13)$$

one obtaines the following values for the bound state energy:

$$E_1 = \frac{1}{2}\Delta \; ; \; E_2 = 2\Delta - 2I\left(1 - \frac{1}{2}\sin^2\frac{Q}{2}\right)$$

We see that the unphysical state with two bosons bound at one lattice point becomes observable in this representation ($E_1 \neq 0$) and this is physically nonsence because it is clear that for spin S = 1/2 two bosons bound at one lattice point create the same physical situation as in the case when that point is not excited.

4. CONCLUSION

It is shown in this paper that the problem of the bound states in ferromagnet with spin S = 1/2 can not be investigated in the spin operator picture but only in the boson representation of the spin operators. The only correct result is obtainable by means of the exact boson representation |4|. Dyson's representation gives good values for the energy but incorrect wave function while in the Marumori's representations we get an additional observable level which has no physical sence.

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TWO-BOSON BOUND STATES IN HEISENBERG FERROMAGNET

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1. Introduction

The problem of the bound states in the Heisenberg ferromagnet with spin $S = 1/2$ has been considered recently in /1/ by the same authors. The reasons to do this analysis were the following:

First, for spin $S = 1/2$, the equation describing the bound states of two spins could be formed in an arbitrary way owing to the fact that the all set of states $S^{-}_{\vec{n}} S^{-}_{\vec{n}} | 0 >$ ($|\vec{n}| = 0,1,2...N$) is equal to zero; e.g. the corresponding coefficients in the wave function describing the bound states may have the arbitrary values. This difficulty can be removed only by introducing some additional assumptions (this question is presented in details in /2/) what, by itself, means that the problem can not be resolved in a selfconsistent way.

Second, since the problem of the bound states is exactly soluble, we used it to test the adequacy of the different boson representation (Dyson, Marumori).

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The results obtained in /1/ show that the exact boson representation of the spin operators and the representation of Dyson give the same values for the bound state's energy but the different wave functions. It has to be pointed out that the twoboson's states and not the twomagnon's states in the boson representation were investigated in /1/ in terms of the exact boson representation (in order to remove any terminological misunderstanding we define the boson states by the wave function $|B\rangle = B^{+}|0\rangle$ where B^{+} is the operator creating Bose quasi-particle and the magnon states by $|S\rangle = S^{-}|0\rangle$ where S^{-} is the spin creation operator). Therefore, in this paper we want to investigate the two-boson and two-magnon processes of the Heisenberg ferromagnet in order to suggest the methods capable of giving the answer to the question what is the nature of real elementary excitations in ferromagnets: are they magnons or bosons.

2. The Energies and the Wave Functions of the Two-Boson Bound States in the Heisenberg Ferromagnet

The spin operator representation of the Hamiltonian of the isotropic Heisenberg ferromagnet of a simple cubic structure has the following form in the nearest neighbour representation:

$$H = E_0 + \Delta \sum_{\vec{n}} (S - S^z_{\vec{n}}) - \frac{I}{2} \sum_{\vec{n},\vec{\lambda}} S^-_{\vec{n}} S^+_{\vec{n}+\vec{\lambda}} - \frac{I}{2} \sum_{\vec{n},\vec{\lambda}} (S - S^z_{\vec{n}})(S - S^z_{\vec{n}+\vec{\lambda}}) \qquad (2.1)$$

where

$$E_0 = -N\,\tilde{\mu}_0 \mathcal{H} S - N S^2 I l$$

$$\Delta = \tilde{\mu}_0 \mathcal{H} + 2 S l I$$

N: the number of atoms in the crystal

l: the number of the dimensions of the crystal

$\vec{\lambda}$: the vector connecting the nearest neighbours

I: the exchange integral for the nearest neighbours

$\mathcal{H}$: the external magnetic field

$\tilde{\mu}_0$: the magnetic moment of the atom; $S^{\pm}_{\vec{n}} = S^{x}_{\vec{n}} \pm i S^{y}_{\vec{n}}$ the spin operators satisfying the relations:

$$\begin{aligned} &[S^{+}_{\vec{n}}, S^{-}_{\vec{n}}] = \delta_{\vec{n}\vec{n}} S^{z}_{\vec{n}} \\ &\{S^{+}_{\vec{n}}, S^{-}_{\vec{n}}\} = 2S(S+1) - 2(S^{z}_{\vec{n}})^2 \\ &(S^{+}_{\vec{n}})^{2S+1} = (S^{-}_{\vec{n}})^{2S+1} = 0 \end{aligned} \tag{2.2}$$

Since we intend to analyse the bound states of two bosons we have to replace the spin operators in (2.1) by the Bose operators, using the exact Bose representation of the spin operators given in /3/. In the case of the low temperatures it is enough to take the following effective boson representation of the spin operators:

$$S^{+}_{\vec{n}} = \sqrt{2S}\,(B_{\vec{n}} - \Omega B^{+}_{\vec{n}} B_{\vec{n}} B_{\vec{n}})$$

$$S^{-}_{\vec{n}} = \sqrt{2S}\,(B^{+}_{\vec{n}} - \Omega B^{+}_{\vec{n}} B^{+}_{\vec{n}} B_{\vec{n}})$$

$$S - S^{z}_{\vec{n}} = \begin{cases} B^{+}_{\vec{n}} B_{\vec{n}} - B^{+}_{\vec{n}} B^{+}_{\vec{n}} B_{\vec{n}} B_{\vec{n}} & ;\ \text{for } S = \frac{1}{2} \\ B^{+}_{\vec{n}} B_{\vec{n}} & ;\ \text{for } S > \frac{1}{2} \end{cases} \tag{2.3}$$

where

$$\Omega = 1 - \sqrt{1 - \frac{1}{2S}} \tag{2.4}$$

In this paper we shall consider the case $S > 1/2$. We shall see later that all results which will be obtained for $S > 1/2$ contain the results of /1/ as a limit when $S \to 1/2$.

After the substitution of (2.3) into (2.1) we get the effective boson Hamiltonian in a following form:

$$H = E_0 + \Delta \sum_{\vec{n}} B^+_{\vec{n}} B_{\vec{n}} - SI \sum_{\vec{n},\vec{\lambda}} B^+_{\vec{n}} B_{\vec{n}+\vec{\lambda}} +$$

$$+ SI\Omega \sum_{\vec{n},\vec{\lambda}} \left(B^+_{\vec{n}} B^+_{\vec{n}} B_{\vec{n}} B_{\vec{n}+\vec{\lambda}} + B^+_{\vec{n}} B^+_{\vec{n}+\vec{\lambda}} B_{\vec{n}+\vec{\lambda}} B_{\vec{n}+\vec{\lambda}} \right) - \tag{2.5}$$

$$- \frac{I}{2} \sum_{\vec{n},\vec{\lambda}} B^+_{\vec{n}} B^+_{\vec{n}+\vec{\lambda}} B_{\vec{n}+\vec{\lambda}} B_{\vec{n}}$$

The wave function describing twoboson processes in the crystal is of the form:

$$|2\rangle_B = \sum_{\vec{f},\vec{g}} A_{\vec{f},\vec{g}} B^+_{\vec{f}} B^+_{\vec{g}} |0\rangle \tag{2.6}$$

where the coefficients $A_{\vec{f},\vec{g}}$ define the probability for two bosons to appear at the lattice points $\vec{f}$ and $\vec{g}$. The coefficients $A_{\vec{f},\vec{g}}$ are symmetrical, $A_{\vec{f},\vec{g}} = A_{\vec{g},\vec{f}}$. The vacuum state is denoted by $|0\rangle$.

If E_2 is the total energy of a system with two boson excitations we obtain the following equation determining the coefficients $A_{\vec{f},\vec{g}}$.

$$\sum_{\vec{f},\vec{g}} A_{\vec{f},\vec{g}} \left\{ E B^+_{\vec{f}} B^+_{\vec{g}} + \left[B^+_{\vec{f}} B^+_{\vec{g}}, H \right] \right\} |0\rangle = 0$$

$$E = E_2 - E_0 \tag{2.7}$$

which reduces to

$$(E-2\Delta)A_{\vec{f},\vec{g}}+SI\sum_{\vec{\lambda}}(A_{\vec{f}+\vec{\lambda},\vec{g}}+A_{\vec{f},\vec{g}+\vec{\lambda}})=$$

$$=SI\Omega\Big[\delta_{\vec{f}\vec{g}}\sum_{\vec{\lambda}}(A_{\vec{f},\vec{g}+\vec{\lambda}}+A_{\vec{f}+\vec{\lambda},\vec{g}})+2\sum_{\vec{\lambda}}\delta_{\vec{f}+\vec{\lambda},\vec{g}}A_{\vec{g}\vec{g}}\Big]- \tag{2.9}$$

$$-I\sum_{\vec{\lambda}}\delta_{\vec{f}+\vec{\lambda},\vec{g}}A_{\vec{f},\vec{g}}$$

After Fourier transformation

$$A_{\vec{f},\vec{g}}=\frac{1}{N^2}\sum_{\vec{K}_1,\vec{K}_2}e^{i\vec{K}_1\vec{f}+i\vec{K}_2\vec{g}}A_{\vec{K}_1,\vec{K}_2} \tag{2.10}$$

and with new coordinates and momenta in the center of mass system

$$\vec{f}-\vec{g}=\vec{r} \qquad \vec{Q}=\vec{K}_1+\vec{K}_2$$

$$\vec{f}+\vec{g}=2\vec{R} \qquad 2\vec{q}=\vec{K}_1-\vec{K}_2 \tag{2.11}$$

one obtains from (2.9) the integral equation with degenerate kernel

$$\Big[E-2\Delta+4SI\sum_{\vec{\lambda}>0}\cos\frac{\vec{\lambda}\vec{Q}}{2}\cos\vec{q}\vec{\lambda}\Big]\alpha_{\vec{Q}}(\vec{q})=$$

$$=\frac{1}{N}\sum_{\vec{q}'}\Big\{4IS\Omega\sum_{\vec{\lambda}>0}\cos\frac{\vec{\lambda}\vec{Q}}{2}\big[\cos\vec{\lambda}\vec{q}+\cos\vec{\lambda}\vec{q}'\big]- \tag{2.12}$$

$$-2I\sum_{\vec{\lambda}>0}\cos\vec{\lambda}\vec{q}\cos\vec{\lambda}\vec{q}'\Big\}\alpha_{\vec{Q}}(\vec{q}')$$

The functions $\alpha_{\vec{Q}}(\vec{q})$ given by

$$\alpha_{\vec{Q}}(\vec{q})=A_{\frac{\vec{Q}}{2}+\vec{q},\frac{\vec{Q}}{2}-\vec{q}} \tag{2.13}$$

satisfy, due to the symmetricity of the coefficients $A_{\vec f,\vec g}$, the relation

$$\alpha_{\vec Q}(-\vec q) = \alpha_{\vec Q}(\vec q) \tag{2.14}$$

which we used in (2.9) and (2.12).

If the energy E is different from the energy of two free boson waves

$$\mathcal{E} = 2\Delta - 4IS\sum_{\lambda>0}\cos\frac{\vec\lambda\vec Q}{2}\cos\vec\lambda\vec q$$

we can divide (2.12) with $E-\mathcal{E}$ and look for the solutions corresponding to $E > 0$, except $E = \mathcal{E}$. By definition, these solutions, if any exist, correspond to the bound states of two bosons. In the further treatment we shall confine ourselves to the one-dimensional case.

The transitions from the summs to integrals gives the equation

$$\alpha_Q(q) = \frac{\Omega\mu\cos q}{\mu\cos q-\beta}C_1(Q) + \frac{\Omega\mu-\frac{1}{2S}\cos q}{\mu\cos q-\beta}C_2(Q) \tag{2.15}$$

where

$$\mu=\cos\frac{Q}{2}\quad;\quad C_1(Q)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\alpha_Q(q')\,dq'$$

$$\beta=\frac{2\Delta-E}{4IS}\qquad C_2(Q)=\frac{1}{2\pi}\int_{-\pi}^{+\pi}\alpha_Q(q')\cos q'\,dq' \tag{2.16}$$

The equation (2.15) has the nontrivial solutions if the determinant D

$$D = \begin{vmatrix} 1 - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{\mu\Omega\cos q}{\mu\cos q - \beta}\,dq & -\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\mu\Omega - \frac{1}{2S}\cos q}{\mu\cos q - \beta}\,dq \\ -\frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\mu\Omega\cos^2 q}{\mu\cos q - \beta}\,dq & 1 - \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\left(\mu\Omega - \frac{1}{2S}\cos q\right)\cos q}{\mu\cos q - \beta}\,dq \end{vmatrix}$$

is equal to zero. This condition gives the bound state´s energy

$$2S\mu^2(\beta^2 - \mu^2)^{1/2} = (\beta - \mu^2)\left[\beta - (\beta^2 - \mu^2)^{1/2}\right] \qquad (2.17)$$

This equation (of the third order in β) is identical with the equation (19.2.4) from /4/ defining the bound state´s energy of two magnons. Thus, there is no difference between the bound state´s energies of two bosons and two magnons. It is easy to verify that this conclusion holds also in the three-dimensional case.

If we solve (2.15) under the condition D = 0, we get for the coefficients $A_{f,g}$

$$A_{f,g} = \frac{1}{N^2}\sum_{Q,q} e^{iQR + iqr}\,\alpha_Q(q) \qquad (2.18)$$

In the nearest - neighbour approximation it follows from (2.18)

$$A_{f,f} = \frac{1}{N}\sum_{Q} e^{ifQ}\,C_1(Q)$$

$$A_{f,f\pm 1} = \frac{1}{N}\sum_{Q} e^{i(f\pm\frac{1}{2})Q}\,C_2(Q) \qquad (2.19)$$

By substitution of (2.19) into (2.6) and after the normalization we find that the bound state's wave function is:

$$|2\rangle_B = \frac{1}{N\sqrt{2}} \sum_{f,Q} \frac{e^{ifQ}}{\left[1+2\left(\frac{\beta}{\mu}\right)^2 G_Q^2\right]^{1/2}} \Big\{ B_f^+ B_f^+ + \frac{\beta}{\mu} G_Q \left[e^{i\frac{Q}{2}} B_f^+ B_{f+1}^+ + e^{-i\frac{Q}{2}} B_f^+ B_{f-1}^+ \right] \Big\} |0\rangle \qquad (2.20)$$

where

$$G_Q = \left[\frac{\beta}{\beta + \mu^2(2S-1)} - \Omega \right]^{-1} \qquad (2.21)$$

It has to be pointed out that for S = 1/2 the wave function (2.20) reduces to the wave function (2.36) from /1/.

3. The Wave Function of the Bound State of two Magnons

The wave function of the twomagnon state has in the case of spin S = 1/2 the following form:

$$|2\rangle_S = \sum_{\vec{f},\vec{g}} D_{\vec{f},\vec{g}} S_{\vec{f}}^- S_{\vec{g}}^- |0\rangle \qquad (3.1)$$

As it is well known this function is by the rule normalized to

$$_S\langle 2 | 2 \rangle_S = 4S^2 \qquad (3.2)$$

In the one-dimensional case we get the following integral equation (see /4/):

$$\gamma_Q(q) = \frac{1}{2S}\,\frac{\cos q}{\mu\cos q - \beta}\,C(Q) \tag{3.3}$$

defining the functions $\gamma_Q(q)$ which figure in the plane--wave expansion of the coefficients D_{fg}

$$D_{fg} = \frac{1}{N^2}\sum_{Q,q}\gamma_Q(q)\,e^{iRQ+irq} \tag{3.4}$$

The function $C(Q)$ is given by

$$C(Q) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}(\mu - \cos q)\,\gamma_Q(q)\,dq \tag{3.5}$$

with $(\mu\cos q - \beta) \neq 0$.

The energies of the bound states are determined with condition

$$1 - \frac{1}{2\pi}\int_{-\pi}^{+\pi}(\mu - \cos q)\,\frac{1}{2S}\,\frac{\cos q}{\mu\cos q - \beta}\,dq = 0 \tag{3.6}$$

which reduces to the equation (2.17) confirming the identity of the bound state's energies of two bosons and two magnons.

Replacing the coefficients $\gamma_Q(q)$ from (3.4) to (3.3) we get in the nearest-neighbour approximation

$$D_{ff} = 2S\,\frac{1}{N}\sum_{Q} e^{iQf}\,C'(Q)$$

$$D_{ff\pm 1} = 2S\,\frac{1}{N}\sum_{Q} e^{iQ(f\pm\frac{1}{2})}\,\frac{\beta}{\mu}\,C'(Q) \tag{3.7}$$

Introducing (3.7) into (3.1) and normalizing according to

(3.2) we obtain the bound state wave function of two magnons in the form

$$|2\rangle_S = \frac{1}{N\sqrt{2}} \sum_{Q,f} \frac{e^{iQf}}{[1+2(\frac{\beta}{\mu})^2]^{1/2}} \{ S_f^- S_f^- + \frac{\beta}{\mu} [e^{i\frac{Q}{2}} S_f^- S_{f+1}^- + e^{-i\frac{Q}{2}} S_f^- S_{f-1}^-] \} |0\rangle \tag{3.8}$$

By the comparaison of (3.8) and (2.20) we see that the bound state wave functions of two magnons and two bosons are different. We have to notice that Dyson's representation of the spin operators

$$S_f^+ = \sqrt{2S} \left(1 - \frac{b_f^+ b_f}{2S}\right) b_f \; ; \; S_f^- = \sqrt{2S}\, b_f^+ \; ; \quad S - S_f^z = b_f^+ b_f \tag{3.9}$$

gives the wave function of the bound states in the following form:

$$|2\rangle_D = \frac{1}{N\sqrt{2}} \sum_{Q,f} \frac{e^{iQf}}{[1+2(\frac{\beta}{\mu})^2]^{1/2}} \{ b_f^+ b_f^+ + \frac{\beta}{\mu} [e^{i\frac{Q}{2}} b_f^+ b_{f+1}^+ + e^{-i\frac{Q}{2}} b_f^+ b_{f-1}^+] \} |0\rangle \tag{3.10}$$

identical with (3.8). For $S = 1/2$ (3.10) reduces to (3.20) from /1/. Concerning the formula (3.8), it reduces also to (3.20) from /1/, but there is an open question whether the term $S_f^- S_f^- |0\rangle$ has to be retained or rejected. As we know for $S = 1/2$ $(S_f^-)^2 = 0$ and that is the reason to reject this term. On the other hand if we begin the consideration with $(S_f^-)^2 = 0$ the equation defining the bound states could not be formed e.g. we are forced in order to obtain this equation, to assume that $(S_f^-)^2 |0\rangle \neq 0$. Therefore one is not able to say anything definitely about the bound state wave function of two magnons.

Concluding this section we shall give the results which can be obtained if we treat the bound states of two magnons in terms of the exact boson representation e.g. with the wave function

$$|2\rangle_{SB} = \sum_{\vec{f},\vec{g}} F_{\vec{f},\vec{g}} (1-\Omega\delta_{\vec{f}\vec{g}}) B^+_{\vec{f}} B^+_{\vec{g}} |0\rangle \tag{3.11}$$

The analysis carried out with (3.11) gives the same value for the bound state energy (see (2.17)) and the wave function which is, in the onedimensional case, of the form

$$|2\rangle_{SB} = \frac{1}{N\sqrt{2}} \sum_{q,f} \frac{e^{iqf}}{\left[1+2\left(\frac{\beta}{\mu}\right)^2 R_\Omega^2\right]^{1/2}} \Big\{ B^+_f B^+_f + \frac{\beta}{\mu} R_\Omega \left[e^{i\frac{q}{2}} B^+_f B^+_{f+1} + e^{-i\frac{q}{2}} B^+_f B^+_{f-1} \right] \Big\} |0\rangle \tag{3.12}$$

where

$$R_\Omega = \frac{1}{1-\Omega}$$

As we see the wave function (3.12) is diferent from (3.8) and (4.20). Only for S = 1/2 it reduces, as (2.20), to (2.36) from /1/.

4. Conclusion

As it was pointed out in the Introduction the basic intention of this paper is to find whether there is a difference between the magnon and boson twoparticle processes in the ferromagnet. The magnon and the boson vacuum as the oneparticle states are identical. In the preceeding two sections it is shown that in the problem of the bound states (which is, let it be mentioned once more, exactly soluble) this difference reflects on the bound state's wave function of two magnons and two bosons.

All evaluations which will be presented in this paper, are carried out for the onedimensional crystal lattice, because it is impossible to get any result in the case of the threedimensional lattice without computer. In this sence and on this level our considerations are only of the academical importance. However, we consider as esential to resolve the question: what is the physical reality in the ferromagnet; the magnons or bosons, and to trace the direction of the further investigations in order to get the answer. The wave functions of two bosons and two magnons are different so it is evident that this difference will play the important role in some physical processes. As in /1/ we shall confine ourselves to the process of the transition from the bound to the free states influenced by the external radiofrequent magnetic field. The transitions probabilities evidently different due to the wave functions, will be evaluated in both pictures, the magnon and the boson one.

We shall calculate the probabilities

$$W_B \sim \left| {}_B\langle 1 | H_{RF} | 2 \rangle_B \right|^2 \tag{4.1}$$

$$W_S \sim |{}_S\langle 1|H_{RF}|2\rangle_S|^2 \tag{4.2}$$

where

$$|1\rangle_S = |1\rangle_B = \frac{1}{\sqrt{2SN}} \cdot \sum e^{ifk} S_f^- |0\rangle = \\ = \frac{1}{\sqrt{N}} \cdot \sum_f e^{ifk} B_f^+ |0\rangle \tag{4.3}$$

and H_{RF} (see /1/):

$$H_{RF} = -\tilde{\mu}_0 \sum_{f,\omega} e^{-i\omega t} \left[\frac{1}{2} h_f^{(-)}(\omega) S_f^+ + \right. \\ \left. + \frac{1}{2} h_f^{(+)}(\omega) S_f^- + h_f^{(z)}(\omega) S_f^z \right] \tag{4.4}$$

where ω are the energies (the field frequencies) and $h_f^{(\pm)}(\omega)$, $h_f^{(z)}(\omega)$ the field components in an energy representation.

To calculate W_S we have to use the formula (2.3). The details of the evaluation are i~~n~~dentical to those presented in /1/. Here as in /1/, we assume that $h^{(-)}(\omega)$ depends weakly on ω, so we ~~may~~ consider it as a constant. We obtained for the transition probabilities the expressions:

$$W_S = 2S\tilde{\mu}_0^2 |h^{(-)}|^2 \int_0^\pi dk \frac{\left[1 - \frac{1}{2S} + 2\beta(\cos\frac{k}{2})\right]^2}{1 + 2\frac{\beta^2(\cos\frac{k}{2})}{\cos^2\frac{k}{2}}} \tag{4.5}$$

$$W_B = 2S\tilde{\mu}_0^2 |h^{(-)}|^2 \int_0^\pi dk \frac{\left[1 - \Omega + 2\beta(\cos\frac{k}{2})\, G(\cos\frac{k}{2})\right]^2}{1 + 2\frac{\beta^2(\cos\frac{k}{2})}{\cos^2\frac{k}{2}} G^2(\cos\frac{k}{2})} \tag{4.6}$$

Since these probabilities may be evaluated numerically, when it would be possible to measure them, we could decide, by the comparaison of the theory and experiment, which is the correct one. This would give also the answer on the question what is the physical reality in the ferromagnet: the spin or the boson fields.

There is no doubt that the corresponding result for the threedimensional lattice could be subjected to the experimental verification. The procedure, presented here, can be simply applied to the three-dimensional lattice; unfortunately it is not posible to give the final results in the closed analitical form but it is necessary to tabulate them with help of the computer.

The ratio W_B/W_S has been found in the case of the spin $S = 1/2$ in /1/ (it was assumed that $S_f^- S_f^- |0\rangle \neq 0$ e.g. that the spin wave functions is identical to the wave function in Dyson-Maljeeff's representation). Here we shall give the values of this ratio for $S = 1, \frac{3}{2}$, 2 and as a limit when $S \to \infty$. The result are given in the tabela:

Table 1.

S	$\frac{1}{2}$	1	$\frac{3}{2}$	2	
$\frac{W_B}{W_S}$ 8	146	105	90	68	54

In this work it is chosen one of the possible processes in the ferromagnet as a test for the question we acked.

Evidently, this is not a only one; probably there are the other processes in which this difference could be greater and more remarkable what would give us the better chances for the experimental verification. This problem will be the subject of our futher investigations.

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SUMMARY

In this paper the two-boson and two-magnon processes in the Heisenberg ferromagnet with the arbitrary value of spin are considered. It is found that for the same energies the bound state's wave functions of two magnons and two bosons are different. On the basis of this there is given the criterium which could give the answer what really exists in the ferromagnet: the boson or the magnon excitations.

Glava treća

Matematički aspekti egzaktne bozonske reprezentacije spinskih operatora

U prve dve glave ove teze detaljno su analizirane dinamičke karakteristike Heisenberg-ovog feromagneta sa spinom $S=\frac{1}{2}$, procesi u feromagnetu uključujući Bose kondenzaciju, zatim vezana stanja i procesi u kojima vezana stanja učestvuju. Zajedničko za ove analize jeste korišćenje egzaktne bozonske reprezentacije spinskih operatora, a osnovna intencija je bila da se uoče vrline /i mane/ koje novi formalizam sobom nosi. Pri tom je naročita pažnja bila upravljena na fizičku stranu problema. Prirodno je, stoga, da, se, posle uspešnih rezultata novog bozonskog formalizma u feromagnetizmu[10), 14)], nelinearnoj optici[8), 11)], i drugim oblastima, posvetimo nekim čisto matematičkim aspektima egzaktne bozonske reprezentacije spinskih operatora.

Kao što je poznato Pauli operatori $P^{+}_{\vec{n}}$ $P_{\vec{n}}$ koji kreiraju odnosno anihiliraju elementarne ekscitacije u kristalu zadovoljavaju sledeće komutacione relacije:

$$\left.\begin{aligned} &P^{+}_{\vec{m}} P_{\vec{n}} - P_{\vec{n}} P^{+}_{\vec{m}} = \left(1-2P^{+}_{\vec{n}}P_{\vec{n}}\right)\delta_{\vec{n}\vec{m}} \\ &P^{+}_{\vec{n}} P^{+}_{\vec{m}} = P^{+}_{\vec{m}} P^{+}_{\vec{n}} ; \quad P_{\vec{n}} P_{\vec{m}} - P_{\vec{m}} P_{\vec{n}} = 0 \\ &\left(P^{+}_{\vec{n}}\right)^2 = \left(P_{\vec{n}}\right)^2 = 0 \end{aligned}\right\} \quad /3.1/$$

Komutacione relacije /3.1/ nisu invarijantne pri prelazu iz prostora direktne u prostor recipročne rešetke a pored toga Pauli operatori nemaju definisanu statistiku što postavlja izuzetne matematičke probleme u cilju izbegavanja gre-šaka u radu sa ovim operatorima. Zbog toga je uobičajen prelazak na operatore Bose ili Fermi tipa koji se transformišu kanonički a imaju određenu statistiku. O prednostima i nedostacima bozonskih reprezentacija poznatih u literaturi bilo je reči u prvoj glavi te ćemo se ovde pozabaviti egzaktnim bozonskom reprezentacijom spinskih operatora.

Egzaktnu bozonsku reprezentaciju Pauli operatori razvili su Agronovič i Tošić[8] baveći se problemima nelinearne optike, pošavši od zahteva da razvoj za Pauli operatore po Bose operatorima zadovoljava komutacione relacije /3.1/. Ona glasi:

$$P^+_{\vec f} = \sqrt{\sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(\nu+1)!} (B^+_{\vec f})^\nu B^\nu_{\vec f}}\; B^+_{\vec f}\, ; \quad P^-_{\vec f} = B_{\vec f} \sqrt{\sum_{\nu=0}^{\infty} \frac{(-2)^\nu}{(\nu+1)!} (B^+_{\vec f})^\nu B^\nu_{\vec f}} \qquad .../3.2a/$$

$$\frac{1}{2} - P^+_{\vec f} P^-_{\vec f} = \hat{A}_{\vec f} \qquad /3.2b/$$

gde je:

$$\hat{A}_{\vec f} = \sum_{\nu=0} \frac{(-2)^\nu}{(\nu+1)!} (B^+_{\vec f})^\nu (B_{\vec f})^\nu \qquad /3.3/$$

Kako se $\sqrt{\hat{A}_{\vec f}}$ pojavljuje u Pauli Hamiltonianima teorije magnetizma i nelinearno optike prirodno se nameće pitanje hermiticiteta ovih Hamiltonijana prepisanih u egzaktnoj bozonskoj slici. Ovaj problem rešen je u radu[24)] koji je priložen u ovoj glavi na sledeći način: Dokazano je da je $\hat{A}_{\vec f}$ pozitivno Semi - definitan operator u celom prostoru bozonskih stanja $|n_{\vec f}\rangle = \frac{1}{\sqrt{n_{\vec f}!}}(B^+_{\vec f})^{n_{\vec f}}|0\rangle$ na taj način što je pokazano da su svojstvene vrednosti $a_{n_{\vec f}}$ operatora $\hat{A}_{\vec f}$ nenegativni brojevi. Ovo je pak dovoljno da Hamiltonijani u kojima figuriše $\sqrt{\hat{A}_{\vec f}}$ budu hermitski.

svojstveni problem $\hat{A}_{\vec f}$ glasi:

$$\hat{A}_{\vec f}\,|n_{\vec f}\rangle = a_{n_{\vec f}}|n_{\vec f}\rangle \qquad /3.4/$$

Operator $(B^+_{\vec f})^{\nu} B^{\nu}_{\vec f}$ može se napisati na sledeći način:

$$(B^+_{\vec f})^{\nu}(B_{\vec f})^{\nu} = \hat{N}_{\vec f}(\hat{N}_{\vec f}-1)\cdot\ldots\cdot(\hat{N}_{\vec f}-\nu+1) \qquad /3.5/$$

gde $\hat{N}_{\vec f} = B^+_{\vec f} B_{\vec f}$ predstavlja operator populacije koji delujući na $|n_{\vec f}\rangle$ daje multiplikator $n_{\vec f}$. Na taj način imamo:

$$\sum_{\nu=0}^{\infty}\frac{(-2)^{\nu}}{(\nu+1)!}(B^+_{\vec f})^{\nu}B^{\nu}_{\vec f}|n_{\vec f}\rangle = \sum_{\nu=0}^{n_{\vec f}}\frac{(-2)^{\nu}}{(\nu+1)!}n_{\vec f}(n_{\vec f}-1)\cdots(n_{\vec f}-\nu+1)|n_{\vec f}\rangle$$

$$= \frac{1}{2}\left[1-(-1)^{n_{\vec f}+1}\right]\frac{1}{n_{\vec f}+1}|n_{\vec f}\rangle \qquad /3.6/$$

$$n_{\vec f} = 0,1,2,3,\ldots$$

tj. svojstvene vrednosti od A $a_{n_{\vec f}} = \frac{1}{2}\left[1-(-1)^{n_{\vec f}+1}\right]$

su jedrake nuli za neparno $n_{\vec f}$ i $\frac{1}{n_{\vec f}+1}$ za parno n .

Na osnovu ovog može se dokazati, da su svojstvene vrednosti operatora $\sum_{\nu=0}^{\infty}\frac{(-2)^{\nu}}{(\nu+1)!}(B_{\vec{f}}^{+})^{\nu+1}B_{\vec{f}}^{\nu+1}$, koji takođe figuriše u pomenutim Hamiltonijanima, nula ili jedan, jer:

$$\sum_{\nu=0}^{\infty}\frac{(-2)^{\nu}}{(\nu+1)!}(B_{\vec{f}}^{+})^{\nu+1}B_{\vec{f}}^{\nu+1}|m_{\vec{f}}\rangle = B_{\vec{f}}^{+}\hat{A}_{\vec{f}}B_{\vec{f}}|m_{\vec{f}}\rangle$$

$$= \frac{1}{2}\left[1-(-1)^{\hat{m}_{\vec{f}}}\right]|m_{\vec{f}}\rangle$$

$$m_{\vec{f}} = 1, 2, 3, 4, \ldots \qquad /3.7/$$

zbog $B_{\vec{f}}|0\rangle = 0$ sledi

$$\sum_{\nu=0}^{\infty}\frac{(-2)^{\nu}}{(\nu+1)!}(B_{\vec{f}}^{+})^{\nu+1}B_{\vec{f}}^{\nu+1}|0\rangle = 0 \quad m_{\vec{f}} = 0 \qquad /3.8/$$

Na taj je pokazano da svojstvena vrednost$\left(\frac{1}{2}\right)$, z komponente spina /3.2b/ $S^{z}_{\vec{f}}$ odgovara bozonskim stanjima sa parnim brojem bozona a$\left(-\frac{1}{2}\right)$ stanjima sa neparnim brojem bozona.

Matematičkim jezikom, spektar $S^{z}_{\vec{f}}$ je u egzaktnoj bozonskoj slici degenerisan.

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Proof of Hermiticity of the Pauli Hamiltonian in Exact Bose Representation

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In the quantum theory of magnetism, nonlinear optics, and ferroelectricity the exact Bose representation (1) of the Pauli operators is often used. In private communications, the authors of the paper (1) were often asked the question on the positive semi-definitness of the operator $\hat{A} = \sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!} B^{+k} B^k$ and consequently on the hermiticity of $\sqrt{\hat{A}}$ which figures in the Pauli Hamiltonians of the mentioned theories, written in the exact Bose representation. For the reason of the rigorous mathematical foundation of the theory it is necessary to answer this question. In this short note we shall prove that $\hat{A}$ is a positive semi-definite operator.

It is enough to prove that the eigenvalues of $\hat{A}$ are positive numbers or zero in whole space of boson states. The eigenvalue problem of $\hat{A}$ can be written as follows:

$$\hat{A}\,|n\rangle = a_n\,|n\rangle\,, \tag{1}$$

where $|n\rangle$ denotes the state with n bosons at one lattice point. We have to demonstrate that $a_n \geq 0$.

The proof can be given directly or by the use of the method of mathematical induction. Here we shall solve the eigenvalue problem (1). Operator $B^{+k} B^k$ may be written in the form

$$B^{+k} B^k = \hat{N}(\hat{N}-1)\,\ldots\,(\hat{N}-k+1)\,, \tag{2}$$

where $\hat{N} = B^+B$ is the population operator which gives the multiplicator n acting on the state $|n\rangle$. So we have

$$\sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!} B^{+k} B^k\,|n\rangle = \sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!}\,\hat{N}(\hat{N}-1)\ldots(\hat{N}-k+1)\,|n\rangle$$

$$= \sum_{k=0}^{n} \frac{(-2)^k}{(k+1)!}\,n(n-1)\ldots(n-k+1)\,\frac{n+1}{n+1}\,|n\rangle$$

$$= \sum_{m=1}^{n+1} \frac{(-2)^{m-1}}{m!}(n+1)n \dots \left[(n+1) - m + 1\right] \frac{|n\rangle}{n+1}$$

$$= -\frac{1}{2}\sum_{m=1}^{n+1} \binom{n+1}{m} (-2)^m \, 1^{n+1-m} \frac{|n\rangle}{n+1} \tag{3}$$

$$= -\frac{1}{2}\left[\sum_{m=0}^{n+1} \binom{n+1}{m} (-2)^m \, 1^{n+1-m} - 1\right] \frac{|n\rangle}{n+1}$$

$$= -\frac{1}{2}\left[(-2+1)^{n+1} - 1\right] \frac{|n\rangle}{n+1}$$

$$= \frac{1}{2}\left[1 - (-1)^{n+1}\right] \frac{1}{n+1} |n\rangle ; \; n = 0, 1, 2, \dots$$

e.g. the eigenvalues $a_n = 2^{-1}\left[1 - (-1)^{n+1}\right](n+1)^{-1}$ are equal to zero for an odd n and $(n+1)^{-1}$ for an even n.

In the paper (1), it is quoted also, without rigorous proof that the only eigenvalues of the operator $\sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!} B^{+k+1} B^{k+1}$ which figures in the Hamiltonians mentioned above, are zero and one. On the basis of the above proof we have

$$\sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!} B^{+k+1} B^{k+1} |n\rangle = B^+ \hat{A} B |n\rangle$$

$$= \sqrt{n} B^+ \hat{A} |n-1\rangle$$

$$= \sqrt{n} \frac{1}{2}\left[1 - (-1)^n\right] \frac{1}{n} B^+ |n-1\rangle \tag{4}$$

$$= \frac{1}{2}\left[1 - (-1)^n\right] |n\rangle ; \; n = 1, 2, 3, \dots ;$$

Due to $B|0\rangle = 0$ it follows for n = 0

$$\sum_{k=0}^{\infty} \frac{(-2)^k}{(k+1)!} B^{+k+1} B^{k+1} |0\rangle = 0. \tag{5}$$

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ZAKLJUČAK

U predloženoj disertaciji testirana je na problemima kvantne teorije magnetizma, po mom mišljenju uspešno, egzaktna bozonska reprezentacija spinskih operatora.

Rezultati koji su izloženi u tezi pokazuju da novi prilaz pored metodoloških prednosti dovodi do drugačije fizičke predstave o elementarnim ekscitacijama u fero magnetu. Naime, u prvoj glavi je pokazano da su elementarne ekscitacije dobro opisane pomoću bozona koji međutim imaju populacioni broj koji se razlikuje od standardnog oblika.

U ovoj je glavi takođe formulisana niskotemperaturna teorija Bose kondenzacije u idealnom feromagnetu sa spinom S=1/2.

U drugoj glavi trasiran je pravac budućeg eksperimentalnog i teorijskog istraživanja prirode elementarnih ekscitacija u fero magnetu.

Treća glava inicira strožije matematičko formulisanje egzaktne bozonske reprezentacije spinskih operatora.

Definitivan sud o vrednosti rezultata i metode daće međutim i pored nesumnjivih pozitivnih indikacija eksperimentalna i teorijska istraživnja koja su u toku.

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